Lecture 8: Space Complexity I

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Outline

1. Space Bounded Computation
2. Configuration Graphs
3. Some Space Complexity Classes
4. PSPACE completeness
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3. Some Space Complexity Classes
4. PSPACE completeness
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We look at space bounded computation where a TM performs its tasks using a restricted number of tape cells. The number of tape cells is a function of the input size.

Only cells used in the read/write tapes count towards the space bound.
Space Bounded Computation

**Definition (Space Bounded Computation): The Class SPACE**

Let $S : \mathbb{N} \to \mathbb{N}$ and $L \subseteq \{0, 1\}^*$. We define $L \in \text{SPACE}(S(n))$ if there is a constant $c$ and a TM $M$ deciding $L$ such that on every input $x \in \{0, 1\}^*$, the total number of locations on the read/write tape that are at some point non-blank during $M$’s execution on $x$ is at most $c \cdot S(|x|)$. 

**Definition (Space Bounded Computation): The Class NSPACE**

In the above definition, replace SPACE with NSPACE and the TM with NDTM.

**Remark**

We will restrict our attention to space bounds $S : \mathbb{N} \to \mathbb{N}$ that are space constructible functions. Intuitively, if $S$ is space constructible, then the machine knows the space bound it is operating under.
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- It makes sense to consider space bounded machines with $S(n) < n$ but not $\text{DTIME}(T(n))$ for $T(n) < n$. 
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- It makes sense to consider space bounded machines with \( S(n) < n \) but not \( \text{DTIME}(T(n)) \) for \( T(n) < n \).
- We will assume \( S(n) > \log n \) since the machine needs to remember the address of the cell currently being read.
- \( \text{DTIME}(S(n)) \subseteq \text{SPACE}(S(n)) \) since a TM can access only one tape cell per step and space can be reused.
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Configuration Graph of a Turing Machine

Graph Specification

- A configuration of a TM $M$ consists of the contents of all non-blank entries of $M$’s tapes, state and head position at a particular point of its execution.
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- For every TM $M$ and input $x \in \{0, 1\}^*$, the configuration graph of $M$ on $x$, denoted as $G_{M,x}$ is a directed graph whose nodes represent the possible configurations that $M$ can reach from $C_s^x$, the start configuration.
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- There is an edge from a configuration $C$ to $C'$ if $C'$ can be reached from $C$ in one step according to $\delta$ of $M$. 
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- There is an edge from a configuration $C$ to $C'$ if $C'$ can be reached from $C$ in one step according to $\delta$ of $M$.

- For a DTM, the outdegree of a node is 1 and for a NDTM, it depends on the number of branches.
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Graph Specification continued....

- $G_{M,x}$ is a DAG.
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- We ensure a single accepting configuration $C_{acc}$ by suitably modifying $M$ to erase all its work tapes before halting.
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Graph Specification continued....

- $G_{M,x}$ is a DAG.
- We ensure a single accepting configuration $C_{acc}$ by suitably modifying $M$ to erase all its work tapes before halting.
- In terms of reachability in graphs, $M$ accepts $x$ iff $\exists$ a directed path in $G_{M,x}$ from $C_{s}^{x}$ to $C_{acc}$. 
Claim about $G_{M,x}$

Let $G_{M,x}$ be the configuration graph of a space-$S(n)$ machine $M$ on some input $x$ of length $n$. Then,
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- Every vertex in $G_{M,x}$ can be described using $c \cdot S(n)$ bits where $c$ is a constant depending on $M$. $G_{M,x}$ has at most $2^{cS(n)}$ nodes.
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Let $G_{M,x}$ be the configuration graph of a space-$S(n)$ machine $M$ on some input $x$ of length $n$. Then,

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- There is an $O(S(n))$-size CNF formula $\varphi_{M,x}$ such that for every two strings $C$ and $C'$, $\varphi(M,x)(C, C') = 1$ if and only if $C$, $C'$ encode two neighboring configurations in $G_{M,x}$.
Claim about \(G_{M,x}\)

Let \(G_{M,x}\) be the configuration graph of a space-\(S(n)\) machine \(M\) on some input \(x\) of length \(n\). Then,

- Every vertex in \(G_{M,x}\) can be described using \(c \cdot S(n)\) bits where \(c\) is a constant depending on \(M\). \(G_{M,x}\) has at most \(2^{cS(n)}\) nodes.
- There is an \(O(S(n))\)-size CNF formula \(\varphi_{M,x}\) such that for every two strings \(C\) and \(C'\), \(\varphi(M,x)(C, C') = 1\) if and only if \(C, C'\) encode two neighboring configurations in \(G_{M,x}\).

Proof

The first part is just about the encoding of the TM. The second part follows a similar pattern as the proof of Cook Levin.
A Theorem relating some classes

**Theorem**

For every space constructible \( S : \mathbb{N} \rightarrow \mathbb{N} \),

\[
\text{DTIME}(S(n)) \subseteq \text{SPACE}(S(n)) \subseteq \text{NSPACE}(S(n)) \subseteq \text{DTIME}(2^{O(S(n))})
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- We have already seen $\text{DTIME}(S(n)) \subseteq \text{SPACE}(S(n))$.
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**Proof**
- We have already seen $\text{DTIME}(S(n)) \subseteq \text{SPACE}(S(n))$.
- Also, it is obvious that $\text{SPACE}(S(n)) \subseteq \text{NSPACE}(S(n))$.
- For proving $\text{NSPACE}(S(n)) \subseteq \text{DTIME}(2^{O(S(n))})$, use the concept of configuration graph. Enumerate over all possible configurations to construct $G_{M,x}$ in $2^{O(S(n))}$ time and check using BFS whether $\exists$ a directed path from $C_s^x$ to $C_{acc}$ in $G_{M,x}$. 
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Some Space Complexity Classes

**Definition**

\[
\text{PSPACE} = \bigcup_{c > 0} \text{SPACE}(n^c).
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The class PSPACE is an analog of the class \( P \).
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$$\text{NPSPACE} = \bigcup_{c>0} \text{NSPACE}(n^c).$$ The class NPSPACE is an analog of the class NP.
Some Space Complexity Classes

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\[ L = \text{SPACE}(\log n). \]
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**Definition**

\[ \text{NL} = \text{NSPACE}(\log n) \]
Some Examples

3SAT ∈ PSPACE
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**Claim**

**NP \( \subseteq \) PSPACE**
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- The TM reuses linear space to evaluate all $2^k$ assignments where $k$ is the number of variables.

Claim

NP ⊆ PSPACE

Proof

Cycle through all possible certificates by reusing polynomial space.
Example

Is the following in L?

Check whether the following language is in L?

\[ \text{EVEN} = \{ x \mid x \text{ has an even number of 1s} \}. \]
Example

Is the following in NL?

Check whether the following language is in NL?

\[ \text{PATH} = \{ <G, s, t> \mid G \text{ is a directed graph in which there is a path from } s \text{ to } t \}. \text{ } G \text{ has } n \text{ nodes.} \]
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Solution: PATH ∈ NL
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Solution: PATH ∈ NL

- A NDTM can make a non-deterministic walk starting at s.
Example

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Solution: \( \text{PATH} \in \text{NL} \)

- A NDTM can make a non-deterministic walk starting at \( s \).
- It maintains the index of the vertex the machine is at and nondeterministically picks the next vertex to visit.
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- The machine accepts iff the walk ends at t in at most n steps.
- If the nondeterministic walk has run for n steps already and t has not been encountered, the machine rejects.
- The work tape needs to hold only O(log n) bits for the number of steps that the walk has been made and the identity of the current vertex.
The role of PATH

**Is PATH in L?**

- This is an open problem and is equivalent to the question whether $L = NL$?
- PATH is to NL what 3SAT is to NP.
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Some Relations

We know \( P \subseteq NP \subseteq \text{PSPACE} \). So, there is a strong belief that \( P \neq \text{PSPACE} \).
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Claim

$P = \text{PSPACE} \implies P = \text{NP}$. 
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Claim
\( P = PSPACE \Rightarrow P = NP \).

Definition
A language \( A \) is PSPACE-hard if for every \( L \in PSPACE \), \( L \leq_P A \). If \( A \in PSPACE \), then \( A \) is PSPACE-complete.
Some Relations
We know $P \subseteq \text{NP} \subseteq \text{PSPACE}$. So, there is a strong belief that $P \neq \text{PSPACE}$.

Claim
$P = \text{PSPACE} \Rightarrow P = \text{NP}$.

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A language $A$ is PSPACE-hard if for every $L \in \text{PSPACE}$, $L \leq_P A$. If $A \in \text{PSPACE}$, then $A$ is PSPACE-complete.

Claim
If any PSPACE-complete language is in $P$, then so is every other language in PSPACE. The contrapositive says that if $\text{PSPACE} \neq P$, then a PSPACE-complete language is not in $P$. 

PSPACE completeness
A Quantified Boolean Formula (QBF) is a boolean formula in which variables are quantified using $\exists$ and $\forall$. 

Examples

Consider the QBF $\forall x \exists y (x \land y) \lor (x \land y)$ over $\{0, 1\}$. The above QBF is TRUE.

What about $\forall x \forall y (x \land y) \lor (x \land y)$?
Quantified Boolean Formula

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- We also specify the universe over which the quantifiers work; in our case it is $\{0, 1\}$.
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- Thus, a QBF has the form $Q_1x_1 Q_2x_2 \ldots Q_nx_n \varphi(x_1, x_2, \ldots, x_n)$ where each $Q_i \in \{\exists, \forall\}$ and $\varphi$ is an unquantified boolean formula.
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- What about negation of the above formula?
Example

We can recast SAT in terms of QBF as $\psi = \exists x_1, \ldots, \exists x_n \varphi(x_1, \ldots, x_n)$ is TRUE.

What about negation of the above formula?

The switch of $\exists$ to $\forall$ in case of SAT gives instances of TAUTOLOGY.
A New Language

**Definition: A New Language TQBF**

The language TQBF is the set of QBFs that are TRUE.
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Theorem
TQBF is PSPACE-complete.
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Proof of TQBF ∈ PSPACE
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Proof of TQBF $\in$ PSPACE
- Let $\psi = Q_1x_1 Q_2x_2 \ldots Q_nx_n \varphi(x_1, x_2, \ldots, x_n)$ be a QBF with $n$ variables; and $|\varphi| = m$. We design a recursive algorithm $A$ to decide the truth of $\psi$. 
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The language TQBF is the set of QBFs that are TRUE.

**Theorem**
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**Proof of TQBF ∈ PSPACE**
- Let $\psi = Q_1x_1Q_2x_2\ldots Q_nx_n\varphi(x_1, x_2, \ldots, x_n)$ be a QBF with $n$ variables; and $|\varphi| = m$. We design a recursive algorithm $A$ to decide the truth of $\psi$.
- **Case I** ($n = 0$): $\psi$ contains only constants (TRUE and/or FALSE) and can be evaluated in $O(m)$ time and space.
Proof of $\text{TQBF} \in \text{PSPACE}$

The Proof Continued ...

- **Case II** ($n > 0$): For $b \in \{0, 1\}$, denote by $\psi|_{(x_1=b)}$, a modification of $\psi$ where $Q_1$ is dropped and all occurrences of $x_1$ are replaced with the constant $b$. 
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- $A$ works as follows:
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- \(A\) works as follows:
  - If \(Q_1 = \exists\), then output 1 iff at least one of \(A(\psi|_{(x_1=0)})\) and \(A(\psi|_{(x_1=1)})\) returns 1.
Proof of $\text{TQBF} \in \text{PSPACE}$

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  - If $Q_1 = \exists$, then output 1 iff at least one of $A(\psi|_{x_1=0})$ and $A(\psi|_{x_1=1})$ returns 1.
  - If $Q_1 = \forall$, then output 1 iff both $A(\psi|_{x_1=0})$ and $A(\psi|_{x_1=1})$ returns 1.
Proof of TQBF $\in$ PSPACE

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- $A$ works as follows:
  - If $Q_1 = \exists$, then output 1 iff at least one of $A(\psi|_{x_1=0})$ and $A(\psi|_{x_1=1})$ returns 1.
  - If $Q_1 = \forall$, then output 1 iff both $A(\psi|_{x_1=0})$ and $A(\psi|_{x_1=1})$ returns 1.

- $A$ returns the correct answer as it just works on the definition of $\exists$ and $\forall$. 

As $A$ is recursive, we need to find a recurrence for the space $S(n, m)$ used by $A$ on $\psi$. 


Proof of TQBF $\in$ PSPACE

The Proof Continued ...

- **Case II** ($n > 0$): For $b \in \{0, 1\}$, denote by $\psi|_{(x_1=b)}$, a modification of $\psi$ where $Q_1$ is dropped and all occurrences of $x_1$ are replaced with the constant $b$.

- $A$ works as follows:
  - If $Q_1 = \exists$, then output 1 iff at least one of $A(\psi|_{(x_1=0)})$ and $A(\psi|_{(x_1=1)})$ returns 1.
  - If $Q_1 = \forall$, then output 1 iff both $A(\psi|_{(x_1=0)})$ and $A(\psi|_{(x_1=1)})$ returns 1.
  - $A$ returns the correct answer as it just works on the definition of $\exists$ and $\forall$.

- As $A$ is recursive, we need to find a recurrence for the space $S(n, m)$ used by $A$ on $\psi$. 

Proof of $\text{TQBF} \in \text{PSPACE}$

The Proof Continued ...

- We now use the fact that space can be reused; i.e. $A(\psi|_{x_1=0})$ and $A(\psi|_{x_1=1})$ can use the same space.
Proof of $\text{TQBF} \in \text{PSPACE}$

The Proof Continued ...

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- After computing $A(\psi|_{x_1=0})$, $A$ needs to retain the single bit output for operating with the output of $A(\psi|_{x_1=1})$ which can now reuse the rest of the space.
Proof of $\text{TQBF} \in \text{PSPACE}$

The Proof Continued ...

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- After computing $A(\psi|_{x_1=0})$, $A$ needs to retain the single bit output for operating with the output of $A(\psi|_{x_1=1})$ which can now reuse the rest of the space.
- Assuming that $A$ uses $O(m)$ space to write $\psi|_{x_1=b}$ for its recursive calls, we get $S(n, m) = S(n - 1, m) + O(m)$, which solves to $S(n, m) = O(nm)$. 