

**Test Codes: SIA (Multiple-choice Type) and  
SIB (Short Answer Type) 2010**

Questions will be set on the following and related topics.

**Algebra:** Sets, operations on sets. Prime numbers, factorization of integers and divisibility. Rational and irrational numbers. Permutations and combinations. Binomial theorem. Logarithms. Theory of quadratic equations. Polynomials and remainder theorem. Arithmetic and geometric progressions. Complex numbers. Algebraic inequalities.

**Geometry:** Plane geometry of class X level. Geometry of 2 dimensions with Cartesian and polar coordinates. Concept of a locus. Equation of a line, angle between two lines, distance from a point to a line. Area of a triangle. Equations of circle, parabola, ellipse and hyperbola and equations of their tangents and normals. Mensuration.

**Trigonometry:** Measures of angles. Trigonometric and inverse trigonometric functions. Trigonometric identities including addition formulæ, solutions of trigonometric equations. Properties of triangles. Heights and distances.

**Calculus:** Functions, one-one functions, onto functions. Limits and continuity. Derivatives and methods of differentiation. Slope of a curve. Tangents and normals. Maxima and minima. Curve-tracing using calculus. Methods of integration, definite and indefinite integrals, evaluation of area using integrals.

**Logical Reasoning.**

**Note.** The actual selection paper will have 30 questions in SIA and 10 questions in SIB. The questions in SIA will be multiple-choice type. All questions in this group will have four options of which *exactly one* option will be correct. The questions in SIB will be short answer type.

## Sample Questions for SIA

The following questions have exactly one correct option. Identify it.

- The last digit of  $(2137)^{754}$  is  
(A) 1                      (B) 3                      (C) 7                      (D) 9.
- The sum of all distinct four digit numbers that can be formed using the digits 1, 2, 3, 4, and 5, each digit appearing at most once, is  
(A) 399900              (B) 399960              (C) 390000              (D) 360000.
- The coefficient of  $a^3b^4c^5$  in the expansion of  $(bc + ca + ab)^6$  is  
(A)  $\frac{12!}{3!4!5!}$               (B)  $\binom{6}{3}3!$               (C) 33                      (D)  $3\binom{6}{3}$ .
- Let  $ABCD$  be a unit square. Four points  $E, F, G$  and  $H$  are chosen on the sides  $AB, BC, CD$  and  $DA$  respectively. The lengths of the sides of the quadrilateral  $EFGH$  are  $\alpha, \beta, \gamma$  and  $\delta$ . Which of the following is always true?  
(A)  $1 \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 2\sqrt{2}$   
(B)  $2\sqrt{2} \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 4\sqrt{2}$   
(C)  $2 \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 4$   
(D)  $\sqrt{2} \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 2 + \sqrt{2}$ .
- The set of all real numbers  $x$  satisfying the inequality  $x^3(x+1)(x-2) \geq 0$  is  
(A) the interval  $[2, \infty)$                                       (B) the interval  $[0, \infty)$   
(C) the interval  $[-1, \infty)$                                       (D) none of the above.
- $z_1, z_2$  are two complex numbers with  $z_2 \neq 0$  and  $z_1 \neq z_2$  and satisfying  $\left| \frac{z_1 + z_2}{z_1 - z_2} \right| = 1$ . Then  $\frac{z_1}{z_2}$  is  
(A) real and negative  
(B) real and positive  
(C) purely imaginary  
(D) none of the above need to be true always.
- Let  $A$  be the fixed point  $(0, 4)$  and  $B$  be a moving point  $(2t, 0)$ . Let  $M$  be the mid-point of  $AB$  and let the perpendicular bisector of  $AB$

meet the  $y$ -axis at  $R$ . The locus of the mid-point  $P$  of  $MR$  is

- (A)  $y + x^2 = 2$  (B)  $x^2 + (y - 2)^2 = 1/4$   
(C)  $(y - 2)^2 - x^2 = 1/4$  (D) none of the above.

8. The sides of a triangle are given to be  $x^2 + x + 1$ ,  $2x + 1$  and  $x^2 - 1$ . Then the largest of the three angles of the triangle is

- (A)  $75^\circ$  (B)  $\left(\frac{x}{x+1}\pi\right)$  radians (C)  $120^\circ$  (D)  $135^\circ$ .

9. If  $A, B, C$  are the angles of a triangle and  $\sin^2 A + \sin^2 B = \sin^2 C$ , then  $C$  is equal to

- (A)  $30^\circ$  (B)  $90^\circ$  (C)  $45^\circ$  (D) none of the above.

10. In the interval  $(-2\pi, 0)$ , the function  $f(x) = \sin\left(\frac{1}{x^3}\right)$

- (A) never changes sign  
(B) changes sign only once  
(C) changes sign more than once, but finitely many times  
(D) changes sign infinitely many times.

11. The limit

$$\lim_{x \rightarrow 0} \frac{(e^x - 1) \tan^2 x}{x^3}$$

- (A) does not exist (B) exists and equals 0  
(C) exists and equals  $2/3$  (D) exists and equals 1.

12. If the function

$$f(x) = \begin{cases} \frac{x^2 - 2x + A}{\sin x} & \text{if } x \neq 0 \\ B & \text{if } x = 0 \end{cases}$$

is continuous at  $x = 0$ , then

- (A)  $A = 0, B = 0$  (B)  $A = 0, B = -2$   
(C)  $A = 1, B = 1$  (D)  $A = 1, B = 0$ .

13. In the triangle  $ABC$ , the angle  $\angle BAC$  is a root of the equation

$$\sqrt{3} \cos x + \sin x = 1/2.$$

Then the triangle  $ABC$  is

- (A) obtuse angled (B) right angled  
(C) acute angled but not equilateral (D) equilateral.

14. The number of maps  $f$  from the set  $\{1, 2, 3\}$  into the set  $\{1, 2, 3, 4, 5\}$  such that  $f(i) \leq f(j)$  whenever  $i < j$  is
- (A) 20                      (B) 50                      (C) 35                      (D) 30.
15. Let  $n$  be a positive integer. Consider a square  $S$  of side  $2n$  units. Divide  $S$  into  $4n^2$  unit squares by drawing  $2n - 1$  horizontal and  $2n - 1$  vertical lines one unit apart. A circle of diameter  $2n - 1$  is drawn with its centre at the intersection of the two diagonals of the square  $S$ . How many of these unit squares contain a portion of the circumference of the circle?
- (A)  $4n - 2$                       (B)  $4n$                       (C)  $8n - 4$                       (D)  $8n - 2$ .
16. A lantern is placed on the ground 100 feet away from a wall. A man six feet tall is walking at a speed of 10 feet/second from the lantern to the nearest point on the wall. When he is midway between the lantern and the wall, the rate of change (in ft./sec.) in the length of his shadow is
- (A) 2.4                      (B) 3                      (C) 3.6                      (D) 12.
17. Let  $n$  be a positive integer. Define
- $$f(x) = \min\{|x - 1|, |x - 2|, \dots, |x - n|\}.$$
- Then  $\int_0^{n+1} f(x)dx$  equals
- (A)  $\frac{(n+4)}{4}$                       (B)  $\frac{(n+3)}{4}$                       (C)  $\frac{(n+2)}{2}$                       (D)  $\frac{(n+2)}{4}$ .
18. Consider three boxes, each containing 10 balls labelled 1, 2, ..., 10. Suppose one ball is drawn from each of the boxes. Denote by  $n_i$ , the label of the ball drawn from the  $i$ -th box,  $i = 1, 2, 3$ . Then the number of ways in which the balls can be chosen such that  $n_1 < n_2 < n_3$  is
- (A) 120                      (B) 130                      (C) 150                      (D) 160.
19. The maximum of the areas of the isosceles triangles with base on the positive  $x$ -axis and which lie below the curve  $y = e^{-x}$  is:
- (A)  $1/e$                       (B) 1                      (C)  $1/2$                       (D)  $e$ .
20. Suppose  $a$ ,  $b$  and  $n$  are positive integers, all greater than one. If  $a^n + b^n$  is prime, what can you say about  $n$ ?
- (A) The integer  $n$  must be 2.  
 (B) The integer  $n$  need not be 2, but must be a power of 2.  
 (C) The integer  $n$  need not be a power of 2, but must be even.  
 (D) None of the above is necessarily true.

21. The value of the integral

$$\int_{\pi/2}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}} dx$$

equals (A) 1 (B)  $\pi$  (C)  $e$  (D) none of these.

22. The value of

$$\sin^{-1} \cot \left[ \sin^{-1} \left\{ \frac{1}{2} \left( 1 - \sqrt{\frac{5}{6}} \right) \right\} + \cos^{-1} \sqrt{\frac{2}{3}} + \sec^{-1} \sqrt{\frac{8}{3}} \right]$$

is

(A) 0 (B)  $\pi/6$  (C)  $\pi/4$  (D)  $\pi/2$ .

23. If  $a_n = \left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right)^2 \left(1 + \frac{3^2}{n^2}\right)^3 \cdots \left(1 + \frac{n^2}{n^2}\right)^n$ , then

$$\lim_{n \rightarrow \infty} a_n^{-1/n^2}$$

is

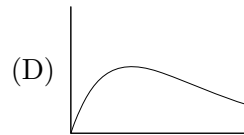
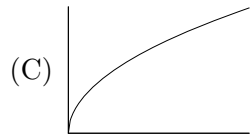
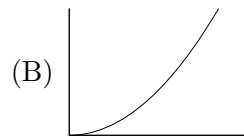
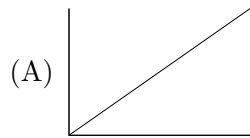
(A) 0 (B) 1 (C)  $e$  (D)  $\sqrt{e}/2$ .

24. If  $f(x) = e^x \sin x$ , then  $\frac{d^{10}}{dx^{10}} f(x) \Big|_{x=0}$  equals

(A) 1 (B) -1 (C) 10 (D) 32.

25. Which of the following graphs represents the function

$$f(x) = \int_0^{\sqrt{x}} e^{-u^2/x} du, \quad \text{for } x > 0 \quad \text{and} \quad f(0) = 0?$$



26. Consider a circle with centre  $O$ . Two chords  $AB$  and  $CD$  extended intersect at a point  $P$  outside the circle. If  $\angle AOC = 43^\circ$  and  $\angle BPD = 18^\circ$ , then the value of  $\angle BOD$  is

(A)  $36^\circ$  (B)  $29^\circ$  (C)  $7^\circ$  (D)  $25^\circ$ .

27. A box contains 10 red cards numbered  $1, \dots, 10$  and 10 black cards numbered  $1, \dots, 10$ . In how many ways can we choose 10 out of the 20 cards so that there are exactly 3 *matches*, where a *match* means a red card and a black card with the same number?

- (A)  $\binom{10}{3} \binom{7}{4} 2^4$  (B)  $\binom{10}{3} \binom{7}{4}$   
 (C)  $\binom{10}{3} 2^7$  (D)  $\binom{10}{3} \binom{14}{4}$

28. The set of complex numbers  $z$  satisfying the equation

$$(3 + 7i)z + (10 - 2i)\bar{z} + 100 = 0$$

represents, in the Argand plane,

- (A) a straight line  
 (B) a pair of intersecting straight lines  
 (C) a pair of distinct parallel straight lines  
 (D) a point
29. The number of triplets  $(a, b, c)$  of integers such that  $a < b < c$  and  $a, b, c$  are sides of a triangle with perimeter 21 is  
 (A) 7 (B) 8 (C) 11 (D) 12.
30. Suppose  $a, b$  and  $c$  are three numbers in G.P. If the equations  $ax^2 + 2bx + c = 0$  and  $dx^2 + 2ex + f = 0$  have a common root, then  $\frac{d}{a}, \frac{e}{b}$  and  $\frac{f}{c}$  are in  
 (A) A.P. (B) G.P. (C) H.P. (D) none of the above.
31. The number of solutions of the equation  $\sin^{-1} x = 2 \tan^{-1} x$  is  
 (A) 1 (B) 2 (C) 3 (D) 5.
32. Let  $\mathbb{R}$  be the set of all real numbers. The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^3 - 3x^2 + 6x - 5$  is  
 (A) one-to-one, but not onto  
 (B) one-to-one and onto  
 (C) onto, but not one-to-one  
 (D) neither one-to-one nor onto.
33. Let  $L$  be the point  $(t, 2)$  and  $M$  be a point on the  $y$ -axis such that  $LM$  has slope  $-t$ . Then the locus of the midpoint of  $LM$ , as  $t$  varies over all real values, is  
 (A)  $y = 2 + 2x^2$  (B)  $y = 1 + x^2$   
 (C)  $y = 2 - 2x^2$  (D)  $y = 1 - x^2$ .
34. Suppose  $x, y \in (0, \pi/2)$  and  $x \neq y$ . Which of the following statements is true?  
 (A)  $2 \sin(x + y) < \sin 2x + \sin 2y$  for all  $x, y$ .  
 (B)  $2 \sin(x + y) > \sin 2x + \sin 2y$  for all  $x, y$ .  
 (C) There exist  $x, y$  such that  $2 \sin(x + y) = \sin 2x + \sin 2y$ .

- (D) None of the above.
35. A triangle  $ABC$  has a fixed base  $BC$ . If  $AB : AC = 1 : 2$ , then the locus of the vertex  $A$  is  
 (A) a circle whose centre is the midpoint of  $BC$   
 (B) a circle whose centre is on the line  $BC$  but not the midpoint of  $BC$   
 (C) a straight line  
 (D) none of the above.
36.  $N$  is a 50 digit number. All the digits except the 26th from the right are 1. If  $N$  is divisible by 13, then the unknown digit is  
 (A) 1                      (B) 3                      (C) 7                      (D) 9.

37. Suppose  $a < b$ . The maximum value of the integral

$$\int_a^b \left( \frac{3}{4} - x - x^2 \right) dx$$

over all possible values of  $a$  and  $b$  is

- (A)  $\frac{3}{4}$                       (B)  $\frac{4}{3}$                       (C)  $\frac{3}{2}$                       (D)  $\frac{2}{3}$ .
38. For any  $n \geq 5$ , the value of  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n - 1}$  lies between  
 (A) 0 and  $\frac{n}{2}$                       (B)  $\frac{n}{2}$  and  $n$   
 (C)  $n$  and  $2n$                       (D) none of the above.

39. In a win-or-lose game, the winner gets 2 points whereas the loser gets 0. *Six* players A, B, C, D, E and F play each other in a preliminary round from which the top *three* players move to the final round. After each player has played *four* games, A has 6 points, B has 8 points and C has 4 points. It is also known that E won against F. In the next set of games D, E and F win their games against A, B and C respectively. If A, B and D move to the final round, the final scores of E and F are, respectively,

- (A) 4 and 2                      (B) 2 and 4                      (C) 2 and 2                      (D) 4 and 4.

40. The number of ways in which one can select six distinct integers from the set  $\{1, 2, 3, \dots, 49\}$ , such that no two consecutive integers are selected, is

- (A)  $\binom{49}{6} - 5\binom{48}{5}$                       (B)  $\binom{43}{6}$   
 (C)  $\binom{25}{6}$                       (D)  $\binom{44}{6}$ .

41. If  $k$  times the sum of the first  $n$  natural numbers is equal to the sum of the squares of the first  $n$  natural numbers, then  $\cos^{-1}\left(\frac{2n-3k}{2}\right)$  is  
 (A)  $\frac{\pi}{6}$                       (B)  $\frac{\pi}{3}$                       (C)  $\frac{2\pi}{3}$                       (D)  $\frac{5\pi}{6}$ .

42. Two circles touch each other at  $P$ . The two common tangents to the circles, none of which pass through  $P$ , meet at  $E$ . They touch the larger circle at  $C$  and  $D$ . The larger circle has radius 3 units and  $CE$  has length 4 units. Then the radius of the smaller circle is  
 (A)  $\frac{1}{2}$                       (B)  $\frac{3}{4}$                       (C)  $\frac{5}{7}$                       (D) 1.

43. Suppose  $ABCDEFGHIJ$  is a ten-digit number, where the digits are all distinct. Moreover,  $A > B > C$  satisfy  $A+B+C = 9$ ,  $D > E > F$  are consecutive even digits and  $G > H > I > J$  are consecutive odd digits. Then  $A$  is  
 (A) 5                      (B) 6                      (C) 7                      (D) 8.

44. Let  $ABC$  be a right angled triangle with  $AB > BC > CA$ . Construct three equilateral triangles  $BCP$ ,  $CQA$  and  $ARB$ , so that  $A$  and  $P$  are on opposite sides of  $BC$ ;  $B$  and  $Q$  are on opposite sides of  $CA$ ;  $C$  and  $R$  are on opposite sides of  $AB$ . Then  
 (A)  $CR = AP = BQ$                       (B)  $CR^2 = AP^2 + BQ^2$   
 (C)  $CR > AP > BQ$                       (D)  $CR < AP < BQ$ .

45. The value of  $(1 + \tan 1^\circ)(1 + \tan 2^\circ) \cdots (1 + \tan 44^\circ)$  is  
 (A) 2                      (B) a multiple of 22  
 (C) not an integer                      (D) a multiple of 4.

46. Let  $y = x/(1+x)$ , where

$$x = \omega^{2009^{2009 \cdots \text{upto } 2009 \text{ times}}}$$

and  $\omega$  is a complex cube root of 1. Then  $y$  is

- (A)  $\omega$                       (B)  $\omega^2$                       (C)  $-\omega$                       (D)  $-\omega^2$ .

47. The number of solutions of  $\theta$  in the interval  $[0, 2\pi]$  satisfying

$$\left(\log_{\sqrt{3}} \tan \theta\right) \sqrt{\log_{\tan \theta} 3 + \log_{\sqrt{3}} 3\sqrt{3}} = -1$$

is

- (A) 6                      (B) 4                      (C) 2                      (D) 0.

48. A building with ten storeys, each storey of height 3 metres, stands on one side of a wide street. From a point on the other side of the street directly opposite to the building, it is observed that the three uppermost storeys together subtend an angle equal to that subtended by the two lowest storeys. The width of the street is
- (A) 6 metres (B)  $6\sqrt{3}$  metres  
(C)  $6\sqrt{35}$  metres (D)  $6\sqrt{70}$  metres.
49. A collection of black and white balls are to be arranged on a straight line, such that each ball has at least one neighbour of different colour. If there are 100 black balls, then the maximum number of white balls that allows such an arrangement is
- (A) 100 (B) 202 (C) 101 (D) 200.
50. Let  $f(x)$  be a real-valued function satisfying  $af(x) + bf(-x) = px^2 + qx + r$ , where  $a$  and  $b$  are distinct real numbers and  $p$ ,  $q$  and  $r$  are non-zero real numbers. Then  $f(x) = 0$  will have real solution when
- (A)  $(\frac{a+b}{a-b})^2 \geq \frac{q^2}{4pr}$  (B)  $(\frac{a+b}{a-b})^2 \geq \frac{4pr}{q^2}$   
(C)  $(\frac{a+b}{a-b})^2 \leq \frac{q^2}{4pr}$  (D)  $(\frac{a+b}{a-b})^2 \leq \frac{4pr}{q^2}$ .
51. A circle is inscribed in a square of side  $x$ , then a square is inscribed in that circle, a circle is inscribed in the latter square, and so on. If  $S_n$  is the sum of the areas of the first  $n$  circles so inscribed, then,  $\lim_{n \rightarrow \infty} S_n$  is
- (A)  $\pi x^2$  (B)  $\frac{\pi x^2}{2}$  (C)  $\frac{\pi x^2}{3}$  (D)  $\frac{\pi x^2}{4}$ .
52. Let  $1, 4, \dots$  and  $9, 14, \dots$  be two arithmetic progressions. Then the number of distinct integers in the collection of first 500 terms of each of the progressions is
- (A) 901 (B) 837 (C) 835 (D) 833.
53. Consider all the 8-letter words that can be formed by arranging the letters in BACHELOR in all possible ways. Any two such words are called *equivalent* if those two words maintain the same relative order of the letters A, E and O. For example, BACOEHLR and CABLROEH are equivalent. How many words are there which are equivalent to BACHELOR?
- (A)  $\binom{8}{3} \times 5!$  (B)  $\binom{8}{3} \times 3!$  (C)  $5! \times 3! \times 2!$  (D)  $2 \times \binom{8}{3}^2$ .

54. The limit

$$\lim_{n \rightarrow \infty} \left( \frac{1}{6} + \frac{1}{24} + \frac{1}{60} + \frac{1}{120} + \cdots + \frac{1}{n^3 - n} \right)$$

equals

- (A)  $\frac{1}{8}$                       (B)  $\frac{1}{4}$                       (C)  $\frac{1}{2}$                       (D) 1.

55. Let  $a$  and  $b$  be real numbers satisfying  $a^2 + b^2 \neq 0$ . Then the set of real numbers  $c$ , such that the equations  $al + bm = c$  and  $l^2 + m^2 = 1$  have real solutions for  $l$  and  $m$  is

- (A)  $(-\infty, \infty)$                       (B)  $[|a + b|, |a + b|]$   
(C)  $[0, a^2 + b^2]$                       (D)  $[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}]$ .

56. Let  $f$  be an onto and differentiable function defined on  $[0, 1]$  to  $[0, T]$ , such that  $f(0) = 0$ . Which of the following statements is necessarily true?

- (A)  $f'(x)$  is greater than or equal to  $T$  for all  $x$   
(B)  $f'(x)$  is greater than or equal to  $T$  for some  $x$   
(C)  $f'(x)$  is smaller than  $T$  for all  $x$   
(D)  $f'(x)$  is smaller than  $T$  for some  $x$ .

57. The area of the region bounded by  $|x| + |y| + |x + y| \leq 2$  is

- (A) 6                      (B) 4                      (C) 3                      (D) 2.

58. Let  $f$  and  $g$  be two positive valued functions defined on  $[-1, 1]$ , such that  $f(-x) = 1/f(x)$  and  $g$  is an even function with  $\int_{-1}^1 g(x)dx = 1$ . Then  $I = \int_{-1}^1 f(x)g(x)dx$  satisfies

- (A)  $I = 1$                       (B)  $I \leq 1$                       (C)  $\frac{1}{3} < I < 3$                       (D)  $I \geq 1$ .

59. How many possible values of  $(a, b, c, d)$ , with  $a, b, c, d$  real, are there such that  $abc = d$ ,  $bcd = a$ ,  $cda = b$  and  $dab = c$ ?

- (A) 1                      (B) 6                      (C) 9                      (D) 17.

60. What is the maximum possible value of a positive integer  $n$ , such that for any choice of seven distinct elements from  $\{1, 2, \dots, n\}$ , there will exist two numbers  $x$  and  $y$  satisfying  $1 < x/y \leq 2$ ?

- (A)  $2^7 - 2$                       (B)  $7^2 - 2$                       (C)  $7^7 - 2$                       (D)  $2 \times 7$ .

## Sample Questions for SIB

1. Consider the function

$$f(x) = \lim_{n \rightarrow \infty} \frac{\log_e(2+x) - x^{2n} \sin x}{1+x^{2n}}$$

defined for  $x > 0$ . Is  $f(x)$  continuous at  $x = 1$ ? Justify your answer. Show that  $f(x)$  does not vanish anywhere in the interval  $0 \leq x \leq \frac{\pi}{2}$ . Indicate the points where  $f(x)$  changes sign.

2. Suppose  $a$  is a complex number such that

$$a^2 + a + \frac{1}{a} + \frac{1}{a^2} + 1 = 0.$$

If  $m$  is a positive integer, find the value of

$$a^{2m} + a^m + \frac{1}{a^m} + \frac{1}{a^{2m}}.$$

3. Let  $f(u)$  be a continuous function and, for any real number  $u$ , let  $[u]$  denote the greatest integer less than or equal to  $u$ . Show that for any  $x > 1$ ,

$$\int_1^x [u]([u] + 1)f(u)du = 2 \sum_{i=1}^{[x]} i \int_i^x f(u)du.$$

4. Two intersecting circles are said to be orthogonal to each other if the tangents to the two circles at any point of intersection are perpendicular to each other. Show that every circle through the points  $(2, 0)$  and  $(-2, 0)$  is orthogonal to the circle  $x^2 + y^2 - 5x + 4 = 0$ .
5. Show that there is exactly one value of  $x$  which satisfies the equation

$$2 \cos^2(x^3 + x) = 2^x + 2^{-x}.$$

6. Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  where  $x_1, \dots, x_n, y_1, \dots, y_n$  are real numbers. We write  $x > y$  if either  $x_1 > y_1$  or for some  $k$ , with  $1 \leq k \leq n-1$ , we have  $x_1 = y_1, \dots, x_k = y_k$ , but  $x_{k+1} > y_{k+1}$ . Show that for  $u = (u_1, \dots, u_n)$ ,  $v = (v_1, \dots, v_n)$ ,  $w = (w_1, \dots, w_n)$  and  $z = (z_1, \dots, z_n)$ , if  $u > v$  and  $w > z$ , then  $u + w > v + z$ .
7. For any positive integer  $n$ , let  $f(n)$  be the remainder obtained on dividing  $n$  by 9. For example,  $f(263) = 2$ .
- (a) Let  $n$  be a three-digit number and  $m$  be the sum of its digits. Show that  $f(m) = f(n)$ .
- (b) Show that  $f(n_1 n_2) = f(f(n_1) \cdot f(n_2))$  where  $n_1, n_2$  are any two positive three-digit integers.

8. Show that it is not possible to have a triangle with sides  $a$ ,  $b$  and  $c$  whose medians have lengths  $\frac{2}{3}a$ ,  $\frac{2}{3}b$  and  $\frac{4}{3}c$ .
9. Let  $P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$  be a polynomial with integer coefficients, such that  $P(0)$  and  $P(1)$  are odd integers. Show that:
- $P(x)$  does not have any even integer as root.
  - $P(x)$  does not have any odd integer as root.
10. Let  $N = \{1, 2, \dots, n\}$  be a set of elements called voters. Let  $\mathcal{C} = \{S : S \subseteq N\}$  be the set of all subsets of  $N$ . Members of  $\mathcal{C}$  are called coalitions. Let  $f$  be a function from  $\mathcal{C}$  to  $\{0, 1\}$ . A coalition  $S \subseteq N$  is said to be *winning* if  $f(S) = 1$ ; it is said to be a *losing* coalition if  $f(S) = 0$ . A pair  $\langle N, f \rangle$  as above is called a voting game if the following conditions hold.
- $N$  is a winning coalition.
  - The empty set  $\emptyset$  is a losing coalition.
  - If  $S$  is a winning coalition and  $S \subseteq S'$ , then  $S'$  is also winning.
  - If both  $S$  and  $S'$  are winning coalitions, then  $S \cap S' \neq \emptyset$ , i.e.,  $S$  and  $S'$  have a common voter.

Show that the maximum number of winning coalitions of a voting game is  $2^{n-1}$ . Find a voting game for which the number of winning coalitions is  $2^{n-1}$ .

11. Suppose  $f$  is a real-valued differentiable function defined on  $[1, \infty)$  with  $f(1) = 1$ . Suppose, moreover, that  $f$  satisfies

$$f'(x) = 1/(x^2 + f^2(x)).$$

Show that  $f(x) \leq 1 + \pi/4$  for every  $x \geq 1$ .

12. Suppose that  $a$  is an irrational number.
- If there is a real number  $b$  such that both  $(a + b)$  and  $ab$  are rational numbers, show that  $a$  is a quadratic surd. ( $a$  is a quadratic surd if it is of the form  $r + \sqrt{s}$  or  $r - \sqrt{s}$  for some rationals  $r$  and  $s$ , where  $s$  is not the square of a rational number).
  - Show that there are two real numbers  $b_1$  and  $b_2$  such that
    - $a + b_1$  is rational but  $ab_1$  is irrational.
    - $a + b_2$  is irrational but  $ab_2$  is rational.
 (Hint: Consider the two cases, where  $a$  is a quadratic surd and  $a$  is not a quadratic surd, separately).
13. (a) Let  $f(x) = x - xe^{-1/x}$ ,  $x > 0$ . Show that  $f(x)$  is an increasing function on  $(0, \infty)$ , and  $\lim_{x \rightarrow \infty} f(x) = 1$ .
- (b) Using part (a) and calculus, sketch the graphs of  $y = x - 1$ ,  $y = x$ ,  $y = x + 1$ , and  $y = xe^{-1/|x|}$  for  $-\infty < x < \infty$  using the same X and Y axes.

14. Find a four digit number  $M$  such that the number  $N = 4 \times M$  has the following properties.
- $N$  is also a four digit number.
  - $N$  has the same digits as in  $M$  but in the reverse order.

15. Consider a function  $f$  on nonnegative integers such that  $f(0) = 1$ ,  $f(1) = 0$  and  $f(n) + f(n-1) = nf(n-1) + (n-1)f(n-2)$  for  $n \geq 2$ . Show that

$$\frac{f(n)}{n!} = \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

16. Suppose  $P$  and  $Q$  are the centres of two disjoint circles  $C_1$  and  $C_2$  respectively, such that  $P$  lies outside  $C_2$  and  $Q$  lies outside  $C_1$ . Two tangents are drawn from the point  $P$  to the circle  $C_2$ , which intersect the circle  $C_1$  at points  $A$  and  $B$ . Similarly, two tangents are drawn from the point  $Q$  to the circle  $C_1$ , which intersect the circle  $C_2$  at points  $M$  and  $N$ . Show that  $AB = MN$ .

17. Evaluate:  $\lim_{n \rightarrow \infty} \frac{1}{2n} \log \binom{2n}{n}$ .

18. Consider the equation  $x^5 + x = 10$ . Show that
- the equation has only one real root;
  - this root lies between 1 and 2;
  - this root must be irrational.

19. In how many ways can you divide the set of eight numbers  $\{2, 3, \dots, 9\}$  into 4 pairs such that no pair of numbers has g.c.d. equal to 2?

20. Suppose  $S$  is the set of all positive integers. For  $a, b \in S$ , define

$$a * b = \frac{\text{l.c.m.}(a, b)}{\text{g.c.d.}(a, b)}$$

For example,  $8 * 12 = 6$ .

Show that **exactly two** of the following three properties are satisfied :

- If  $a, b \in S$  then  $a * b \in S$ .
  - $(a * b) * c = a * (b * c)$  for all  $a, b, c \in S$ .
  - There exists an element  $i \in S$  such that  $a * i = a$  for all  $a \in S$ .
21. Two train lines intersect each other at a junction at an acute angle  $\theta$ . A train is passing along one of the two lines. When the front of the train is at the junction, the train subtends an angle  $\alpha$  at a station on the other line. It subtends an angle  $\beta$  ( $< \alpha$ ) at the same station, when its rear is at the junction. Show that

$$\tan \theta = \frac{2 \sin \alpha \sin \beta}{\sin(\alpha - \beta)}.$$

22. Let  $f(x)$  be a continuous function, whose first and second derivatives are continuous on  $[0, 2\pi]$  and  $f''(x) \geq 0$  for all  $x$  in  $[0, 2\pi]$ . Show that

$$\int_0^{2\pi} f(x) \cos x \, dx \geq 0.$$

23. Let  $ABC$  be a right-angled triangle with  $BC = AC = 1$ . Let  $P$  be any point on  $AB$ . Draw perpendiculars  $PQ$  and  $PR$  on  $AC$  and  $BC$  respectively from  $P$ . Define  $M$  to be the maximum of the areas of  $BPR$ ,  $APQ$  and  $PQCR$ . Find the minimum possible value of  $M$ .
24. A sequence is called an *arithmetic progression of the first order* if the differences of the successive terms are constant. It is called an *arithmetic progression of the second order* if the differences of the successive terms form an arithmetic progression of the first order. In general, for  $k \geq 2$ , a sequence is called an *arithmetic progression of the  $k$ -th order* if the differences of the successive terms form an arithmetic progression of the  $(k - 1)$ -th order. The numbers

$$4, 6, 13, 27, 50, 84$$

are the first six terms of an arithmetic progression of some order. What is its least possible order? Find a formula for the  $n$ -th term of this progression.

25. A cardboard box in the shape of a rectangular parallelepiped is to be enclosed in a cylindrical container with a hemispherical lid. If the total height of the container from the base to the top of the lid is 60 centimetres and its base has radius 30 centimetres, find the volume of the largest box that can be completely enclosed inside the container with the lid on.
26. Let  $f(x)$  be the function satisfying

$$xf(x) = \log x, \quad \text{for } x > 0.$$

Show that  $f^{(n)}(1) = (-1)^{n+1}n! \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ , where  $f^{(n)}(x)$  denotes the  $n$ -th derivative of the function  $f$  evaluated at  $x$ .

27. Show that the vertices of a regular pentagon are concyclic. If the length of each side of the pentagon is  $x$ , show that the radius of the circumcircle is  $\frac{x}{2} \operatorname{cosec} 36^\circ$ .
28. Find the number of ways in which three numbers can be selected from the set  $\{1, 2, \dots, 4n\}$ , such that the sum of the three selected numbers is divisible by 4.
29. Consider 6 points located at  $P_0 = (0, 0)$ ,  $P_1 = (0, 4)$ ,  $P_2 = (4, 0)$ ,  $P_3 = (-2, -2)$ ,  $P_4 = (3, 3)$  and  $P_5 = (5, 5)$ . Let  $R$  be the region consisting of *all* points in the plane whose distance from  $P_0$  is smaller

than that from any other  $P_i$ ,  $i = 1, 2, 3, 4, 5$ . Find the perimeter of the region  $R$ .

30. Let  $x_n$  be the  $n$ -th non-square positive integer. Thus,  $x_1 = 2$ ,  $x_2 = 3$ ,  $x_3 = 5$ ,  $x_4 = 6$ , etc. For a positive real number  $x$ , denote the integer closest to it by  $\langle x \rangle$ . If  $x = m + 0.5$ , where  $m$  is an integer, then define  $\langle x \rangle = m$ . For example,  $\langle 1.2 \rangle = 1$ ,  $\langle 2.8 \rangle = 3$ ,  $\langle 3.5 \rangle = 3$ . Show that  $x_n = n + \langle \sqrt{n} \rangle$ .