Test Code: RC (Short Answer Type) 2005

JRF in Computer and Communication Sciences

The Candidates for Junior Research Fellowship in Computer Science and Communication Sciences will have to take two tests - Test MIII (objective type) in the forenoon session and Test RC (short answer type) in the afternoon session.

In the RC test, there will be five questions from each of these areas :

(i) Mathematics, (ii) Statistics, (iii) Physics at B.Sc./ M.Sc. level, (iv) Radiophysics/ Telecommunication/ Electronics/ Electrical Engg., and (v) Computer Science at B.E./ B. Tech./ M.Sc./ M.E./ M.Tech. level.

The candidates are required to answer any five questions irrespective of the groups.

The syllabi and sample questions of the RC test are as follows.

Syllabus

Mathematics: Graph Theory and Combinatorics, Linear Programming, Linear Algebra, Abstract Algebra, Elementary Number Theory, Calculus (including Ordinary & Partial Differential Equations) and Real Analysis, Integral Transforms.

Statistics: Basic Probability Theory, Distributions and Characteristic Functions, Markov Chains, Estimation and Inference, Linear Models, Multivariate Analysis.

Physics: Classical and Relativistic Mechanics, Heat and Thermodynamics, Non-relativistic Quantum Mechanics, Electricity and Magnetism, d.c. and a.c. Circuits, Basics of Semiconductor Physics, Vibration and Waves.

Radiophysics/Telecommunication/Electronics/Electrical Engg.: Boolean Algebra, Digital Circuits and Systems, Circuit Theory, Amplifiers, Oscillators, Digital Communication, Digital Signal Processing, Electrical Machines.

Computer Science: Data Structures, Sorting/Searching, Design and Analysis of Algorithms, Computer Architecture, Operating Systems, Automata and Formal Languages, Principles of Compiler Construction, Computer Networks, Databases.

Sample Questions

Note that all questions are **not** *of equal weight*

(i) MATHEMATICS

1. (a) Let $f : I\!\!R \to I\!\!R$ be a differentiable function for which there does not exist any $x \in [0, 1]$ such that both $f(x) = 0$ and $f'(x) = 0$. Show that f has only a finite number of zeros in $[0, 1]$.

   [$f'(x)$ denotes the derivative of f at x].

   (b) Let $f : I\!\!R \to I\!\!R$ be such that $f'(x)$ exists and is continuous in $[0, \infty)$. Show that

$$\lim_{x \downarrow 0} \frac{1}{x^2} \int_0^x (x - 3y) f(y) dy = -\frac{f(0)}{2}.$$

   [$x \downarrow 0$ denotes: x decreases to zero.]

2. (a) If $f : I\!\!R \to I\!\!R$ is twice differentiable, prove that:

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

   (b) Let $f : [0, 1] \to I\!\!R$ be a function such that f is Riemann integrable over $[b, 1]$ for all b such that $0 < b \leq 1$.

   i) If f is bounded, prove that f is Riemann integrable over $[0, 1]$.

   ii) What if f is not bounded?

3. (a) If u is a function of x, y and z satisfying the partial differential equation

$$(y - z)\frac{\partial u}{\partial x} + (z - x)\frac{\partial u}{\partial y} + (x - y)\frac{\partial u}{\partial z} = 0.$$

   Show that u is of the form, $u = \psi(x + y + z,\ x^2 + y^2 + z^2)$, for some function ψ.

   (b) Prove that

$$(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$$

   represents hyperbolas having the following lines as asymptotes:

$$x + y = 0,\ 2x + y = 0.$$

4. (a) Find the Fourier transform of the unit impulse (Dirac's delta function $\delta(t)$).

(b) Find the Laplace transform of the periodic function $f(t)$ defined by

$$f(t) = (t - nT)^2, for \ \ nT \leq t \leq (n+1)T, \ n \geq 0, \ T > 0.$$

5. (a) Let A be a real $n \times n$ matrix. Show that if $A^2 = I$ then each eigen value of A is 1 or -1. Is the converse true? Justify your answer.

   (b) Does there exist a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $f(\underline{x}_1) = \underline{y}_1$, $f(\underline{x}_2) = \underline{y}_2$ and $f(\underline{x}_3) = \underline{y}_3$ where $\underline{x}_1^t = (0,1,1)$, $\underline{x}_2^t = (1,1,0)$, $\underline{x}_3^t = (3,5,2)$, $\underline{y}_1^t = (1,0,0)$, $\underline{y}_2^t = (0,1,0)$ and $\underline{y}_3^t = (0,0,1)$? Justify your answer. [$t$ denotes 'transpose']

6. (a) Show that the group of all positive rational numbers under multiplication is isomorphic to the additive group $\mathbf{Z}[X]$ of all polynomials in X with integer coefficients.

   [Hint : Think of prime factorization.]

   (b) Show that there does not exist any nontrivial homomorphism from the additive group of real numbers to the additive group of integers.

7. (a) Prove that any finitely generated subgroup of $(Q, +)$ is cyclic, where $(Q, +)$ is the group of rational numbers with usual addition operation $+$.

   (b) Prove that $Aut(Q, +) \cong Z_2$ where Z_2 is the group consisting of only two elements; and $Aut(Q, +)$ is the automorphism group of $(Q, +)$.

8. (a) Solve the following linear programming problem

$$\begin{array}{ll} Maximize & Z = 2x_1 + x_2 \\ subject \ to & 2x_1 + 5x_2 \leq 17 \\ & 3x_1 + 2x_2 \leq 10 \\ & x_i \geq 0 \ \ \forall i \end{array}$$

   (b) Does the above solution change if the condition '$x_i \geq 0 \ \forall i$' is removed?

9. Let each row and each column of a $n \times n$ matrix A be a permutation of $\{1, 2, ..., n\}$ and let A be symmetric.

   (a) If n is odd, prove that each of $1, 2, ...n$ occurs on the principle diagonal of A.

   (b) For every even number n, show that there exists an A in which not all of $1, 2, ..., n$ appear on the diagonal.

10. Let k be a positive integer. Let $G = (V, E)$ be the graph where V is the set of all strings of 0's and 1's of length k, and $E = \{(x, y) : x, y \in V,\ x$ and y differ in exactly one place$\}$.
(i) Determine the number of edges in G.
(ii) Prove that G has no odd cycle.
(iii) Prove that G has a perfect matching.
(iv) Determine the maximum size of an independent set in G.

11. (a) Show that, given $2^n + 1$ points with integer coordinates in R^n, there exists a pair of points among them such that all the coordinates of the midpoint of the line segment joining them are integers.

(b) Let $G = (V, E)$ be the complete graph on n-vertices. If n is of the form $3k$, find three pairwise disjoint subsets E_1, E_2, E_3 of E such that $E_1 \cup E_2 \cup E_3 = E$ and the graphs $(V, E_1), (V, E_2), (V, E_3)$ are isomorphic. (You need not prove that they are isomorphic. You may show E_1, E_2, E_3 in a diagram.) If n is of the form $3k + 2$ show that E_1, E_2, E_3 cannot exist as above.

12. Let $\phi(n)$ denote the number of positive integers m relatively prime to n; $m < n$.

(i) Let $n = pq$ where p and q are prime numbers. Then show that

$$\phi(n) = (p-1)(q-1) = pq\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{p}\right).$$

(ii) Show that for any positive integer n

$$\phi(n) = n \prod_{i=1}^{K}\left(1 - \frac{1}{p_i}\right)$$

where p_1, p_2, p_3, …, p_K are distinct prime factors of n.

(ii) STATISTICS

13. (a) Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables satisfying $X_{n+1} = X_n + Z_n$ (addition is modulo 5), where $\{Z_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables with common distribution
$P(Z_n = 0) = 1/2,\ P(Z_n = -1) = P(Z_n = +1) = 1/4.$
Assume that X_1 is a constant belonging to $\{0, 1, 2, 3, 4\}$. What happens to the distribution of X_n as $n \to \infty$?

(b) Let $\{Y_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables with a common uniform distribution on $\{1, 2, ..., m\}$. Define a sequence of random variables $\{X_n\}_{n \geq 1}$ as $X_{n+1} = MAX\{X_n, Y_n\}$ where X_1 is a constant belonging to $\{1, 2, ..., m\}$. Show that $\{X_n\}_{n \geq 1}$ is a Markov chain and classify its states.

14. Let there be r red balls and b black balls in a box. One ball is removed at random from the box. In the next stage $(a + 1)$ balls of the color same as that of the removed ball were put into the box $(a \geq 1)$. This process was repeated n times. Let X_n denote the total number of red balls at the n-th instant.

    (a) Compute $E(X_n)$.

    (b) Show that if $(r + b)$ is much larger than a and n,

$$\frac{1}{r}E(X_n) = (1 + \frac{na}{r + b}) + O(\frac{1}{r + b}).$$

15. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the gamma distribution with density function

$$f(x, \theta) = \frac{\theta^k}{\Gamma(k)} e^{-\theta x} x^{k-1}, \quad 0 < x < \infty,$$

where θ is unknown and $k > 0$ is known. Find a minimum variance unbiased estimator for $\frac{1}{\theta}$.

16. Let $0 < p < 1$ and $b > 0$. Toss a coin once where the probability of occurrence of head is p. If head appears, then n independent and identically distributed observations are generated from Uniform$(0, b)$ distribution. If the outcome is tail, then n independent and identically distributed observations are generated from Uniform$(2b, 3b)$ distribution. Suppose you are given these n observations $X_1, ..., X_n$, but not the outcome of the toss. Find the maximum likelihood estimator of b based on $X_1, ..., X_n$. What happens to the estimator as n goes to ∞ ?

17. Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with common density function f. Define the random variable N as

$$N = n, \text{ if } X_1 \geq X_2 \geq \ldots \geq X_{n-1} < X_n; \text{ for } n = 2, 3, 4, \ldots$$

Find $Prob(N = n)$. Find the mean and variance of N.

18. (a) Let X and Y be two random variables such that

$$\begin{pmatrix} logX \\ logY \end{pmatrix} \sim N(\mu, \sum).$$

Find a formula for $\varphi(t, r) = E(X^t Y^r)$, where t and r are real numbers, and E denotes the expectation.

(b) Consider the linear model
$y_{n\times 1} = A_{n\times p}\beta_{p\times 1} + \varepsilon_{n\times 1}$ and the usual Gauss-Markov set up where, $E(\varepsilon) = 0$ and $D(\varepsilon) = \sigma^2 I_{n\times n}$, E denotes *Expectation* and D denotes *dispersion*.
Assume that A has full rank. Show that $Var(\beta_1^{LS}) = (\alpha - \Gamma^T B^{-1}\Gamma)^{-1}\sigma^2$ where

$$A^T A = \begin{bmatrix} \alpha_{1\times 1} & \Gamma^T_{1\times \overline{p-1}} \\ & \\ \Gamma_{\overline{p-1}\times 1} & B_{\overline{p-1}\times\overline{p-1}} \end{bmatrix}$$

and β_1^{LS} = the least square estimate of β_1, the first component of the vector β. Var denotes the *variance*.

19. Let $p_1(x)$ and $p_2(x)$ denote the probability density functions for classes 1 and 2 respectively. Let P and $(1 - P)$ be the prior probabilities of the classes 1 and 2, respectively. Consider

$$\begin{aligned} p_1(x) &= x && \text{for } x \in [0, 1) \\ &= 2 - x && \text{for } x \in [1, 2] \\ &= 0 && \text{otherwise;} \end{aligned}$$

and

$$\begin{aligned} p_2(x) &= x - 1 && \text{for } x \in [1, 2) \\ &= 3 - x && \text{for } x \in [2, 3] \\ &= 0 && \text{otherwise.} \end{aligned}$$

(i) Find the optimal Bayes risk for this classification problem.

(ii) For which values of P, is the above risk

(I) minimized ?

(II) maximized ?

20. Let $\mathbf{X} = (X_1, ...X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ be two independent and identically distributed multivariate random vectors with mean $\mathbf{0}$ and covariance matrix $\sigma^2\mathbf{I_n}$, where $\sigma^2 > 0$ and $\mathbf{I_n}$ is the $n \times n$ identity matrix.

(a) Show that $\frac{\mathbf{X}^{\mathbf{T}}\mathbf{Y}}{\|\mathbf{X}\|\ \|\mathbf{Y}\|}$ and $V = \sum(X_i^2 + Y_i^2)$ are independent. (Here, $\|(a_1, ..., a_n)\| = \sqrt{a_1^2 + ... + a_n^2}$).

(b) Obtain the probability density of $\dfrac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} Y_i^2}$.

21. Let $X_1, X_2, \ldots, X_n$ be independent random variables. Let $E(X_j) = j\theta$ and $V(X_j) = j^3\sigma^2$, $j = 1, 2, \ldots, n, -\infty < \theta < \infty$ and $\sigma^2 > 0$. Here $E(X)$ denotes expectation and $V(X)$ denotes the variance of the random variable X. It is assumed that θ and σ^2 are unknown.
(i)Find the best linear unbiased estimator for θ.
(ii) Find the uniformly minimum variance unbiased estimate for θ under the assumption that X_i's are normally distributed; $1 \leq i \leq n$.

22. A hardware store wishes to order Christmas tree lights for sale during Christmas season. On the basis of past experience, they feel that the demand v for lights can be approximately described by the probability density function $f(v)$. On each light ordered and sold they make a profit of a cents, and on each light ordered but not sold they sustain a loss of b cents. Show that the number of lights they should order to maximize the expected profit is given by x, which is the solution of the equation:

$$\int_0^x f(v)dv = \frac{a}{a+b}$$

23. Let (X, Y) follow the bivariate normal distribution. Let *mean* of X = *mean* of $Y = 0$. Let *variance* of X = *variance* of $Y = 1$, and the *correlation coefficient* between X and Y be ρ. Find the correlatin coefficient between X^3 and Y^3.

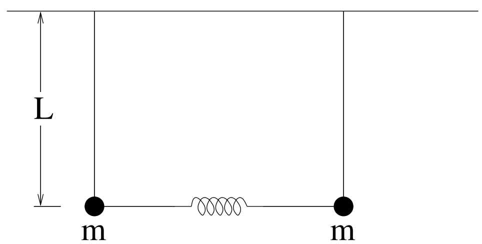
(iii) PHYSICS

24. (a) Calculate the density of donor atoms which has to be added to an intrinsic semiconductor to produce n-type material of resistivity 0.2 *ohm-cm*. It is given that the mobility of electrons in the n-type semiconductor is 2500 $cm^2 volt^{-1} sec^{-1}$.

(b) An n-type semiconductor specimen has a donor density of $10^{15}/cc$. It is arranged in a Hall effect experiment where the magnetic induction field $B = 1.6\ weber/m^2$ and current density $J = 500\ amp/m^2$. What is the Hall Voltage if the specimen is $8/\pi\ mm$ thick?

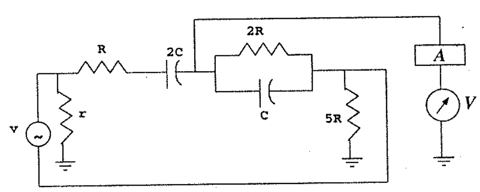
25. In studying the emission of electrons from metals it is necessary to take into account the fact that electrons with energy sufficient to escape from the metal can, according to quantum mechanics, undergo reflection at the surface of the metal. Consider a one dimensional model with the potential $V = -V_0$ for $x < 0$ (inside the metal), and $V = 0$ for $x > 0$ (outside the metal), and determine the reflection coefficient of an electron of energy $E > 0$ at the surface of the metal.

26. Two simple pendulums, each of length L and mass m, are connected by a weightless spring as shown in the figure below. In the equilibrium position the spring is not deformed. Find the frequency of small oscillations when they are deflected in the same plane through the same angle in opposite directions.



27. Consider the circuit shown in figure below. The input supply v is a variable frequency voltage source. A is a high precision ac amplifier whose output is connected to a voltmeter V.

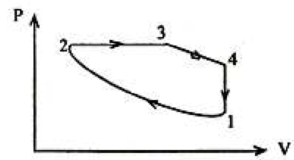
Find the value of r and the angular frequency ω of the input, at which the voltmeter reading would be zero. Assume that $R = 1000\Omega$ and $c = 0.01\mu F$.



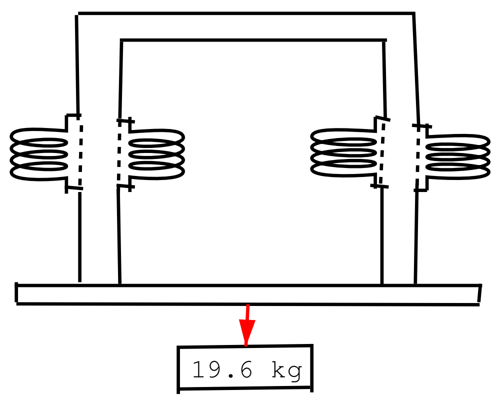
28. Let $\vec{E}$ and $\vec{B}$ be the electric field and the magnetic induction field, respectively at a certain point in space and time in a system K, and let $\vec{E}'$ and $\vec{B}'$ be the corresponding fields at the same point in space but in another system K', moving relative to the system K at a velocity v directed along the X-axis. Write down the expressions for $\vec{E}'$ and $\vec{B}'$ in terms of $\vec{E}$ and $\vec{B}$. Show also that $\vec{E}.\vec{B}$ and $E^2 - c^2B^2$ remain invariant under the Lorentz transformation.

29. Calculate the resultant of two rectangular simple harmonic vibrations whose amplitudes as well as periods are in the ratio 2:1, and the phase difference is 90^0.

30. Calculate the efficiency of the cycle shown in figure below consisting of two adiabats $1 \rightarrow 2$ and $3 \rightarrow 4$; one isobar $2 \rightarrow 3$; and one constant volume process $4 \rightarrow 1$. Assume C_v and C_p are constants.



31. A horse-shoe magnet is formed out of a bar of wrought iron of 50 cm length having cross section 6.28 cm^2. Exciting coils of 500 turns are placed on each limb and connected in series. Find the exciting current necessary for the magnet to lift a load of 19.6 kg (see the figure given below) assuming that the load has negligible reluctance and makes close contact with the magnet. Relative permeability of iron is 700.



32. (a) Find the minimum attainable pressure of an ideal gas in the process $T = T_0 + \alpha V^2$, T_0, α being constants and V is the volume of one mole of the gas.

    (b) A muon of mass 100 Mev/c^2 is produced at an altitude of 10^4 m measured from the earth's surface. How energetic the muon must be in order to reach the earth's surface? The lifetime of muon at rest is 10^{-6} sec.

33. Consider a particle of mass m and energy E approaching a potential

barrier V where

$$V = 0 \text{ for } x < 0; \\ = V_0 \text{ for } 0 \leq x \leq d; \\ = 0; \text{ for } x > d.$$

Show that the transmission co-efficient, T, is given approximately by $T \simeq exp(-2d\sqrt{\frac{2m(V-E)}{(\frac{h}{2\pi})^2}})$. (Assume $d\sqrt{\frac{2m(V-E)}{(\frac{h}{2\pi})^2}} >> 1$.)

34. A solid sphere of weight W rolls without sliding down a plane inclined at an angel of θ to the horizontal. Write down the equations of motion and show that the acceleration of the center of gravity of the body is given by

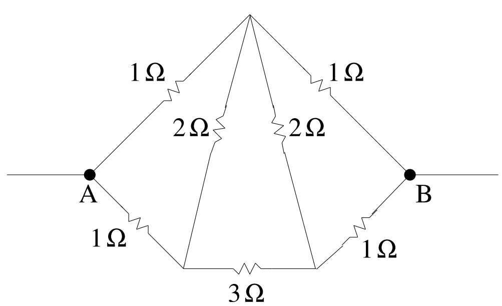
$$a = \frac{g\ sin\theta}{1 + \frac{k^2}{r^2}}$$

where g is the acceleration due to gravity, k is the radius of gyration and r is the radius of the sphere.

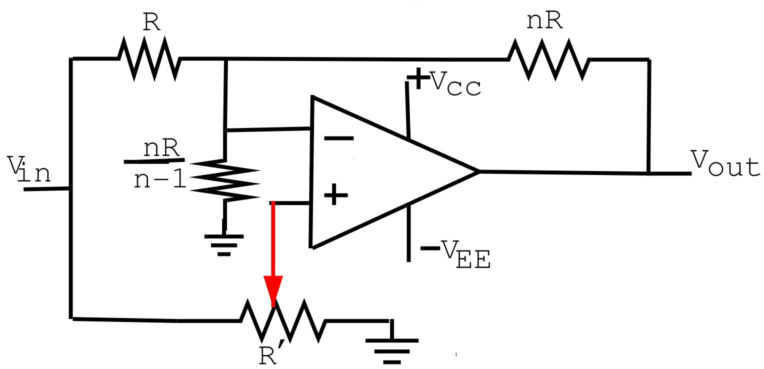
(iv) RADIOPHYSICS/TELECOMMUNICATIONS/ELECTRONICS/ ELECTRICAL ENGG.

35. Design a sequential machine that produces an output 1 whenever a substring of 5 consecutive symbols in the input starts with two 1's and contains exactly three 1's. If a substring of 5 symbols starts with two 1's, the analysis of the next substring does not begin until the processing of the current substring is complete. Realize this circuit with minimum number of NAND gates and flip flops.

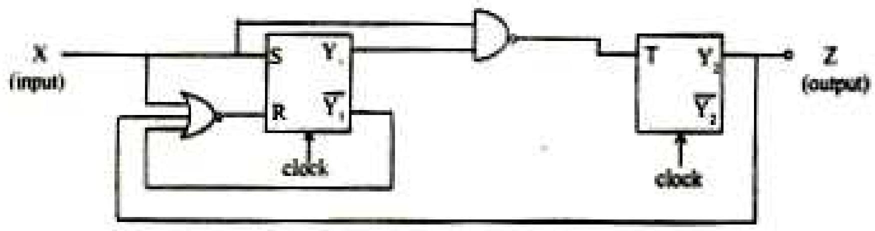
36. A network of resistances is shown in the following figure. Find the equivalent resistance between the points A and B.



37. Calculate the range of voltage gain of the following circuit when the variable resistance R' changes from minimum to maximum.



38. Draw the state table for the synchronous sequential circuit shown in the figure below:

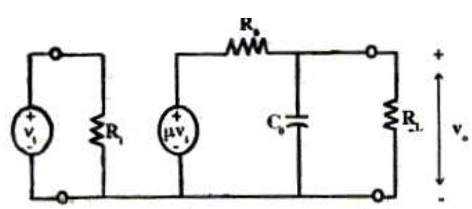


39. Consider a voltage amplifier circuit shown in figure below, where R_i and R_0 represent the input and output impedances respectively, C_0 is the total parasitic capacitance across the output port, μ is the amplifier gain and the output is terminated by a load resistance R_L.
(i) Calculate the current, voltage and power gain in decibels (dB) of the amplifier, when

$R_i = 1M\Omega;\quad R_L = 600\Omega;\quad R_o = 100M\Omega,$
$C_o = 10pf;\quad \mu = 10.$

(ii)Calculate 3-dB cutoff frequency of the amplifier when

$R_i = 5K\Omega;\quad R_L = 1K\Omega;\quad R_o = 100\Omega$
$C_o = 10pf;\quad \mu = 2.$



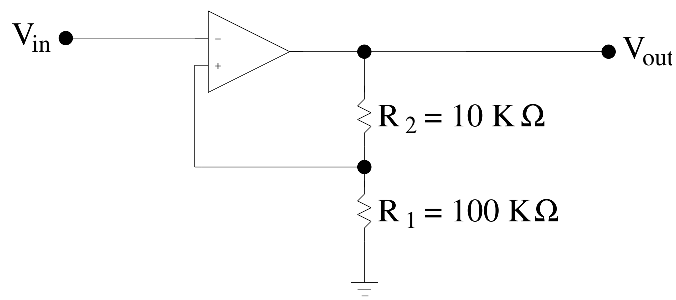
40. (a) Find the Fourier transform of the point spread function

$$h(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

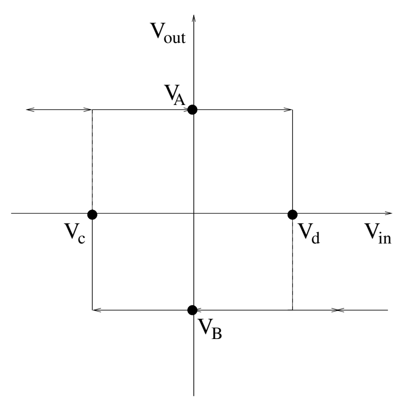
and show that it is rotationally symmetric.

(b) Show that if a signal is passed through the above filter function, high frequencies will be more attenuated in amplitude compared to low frequencies.

41. Consider the following circuit with an OP-AMP.



The plot of output voltage V_{out} vs. input voltage V_{in} for the given circuit is as follows.



Let $V_A = 10\ V$ and $V_B = -10\ V$. Assume that $V_{in} < V_c$, and is gradually increasing. The output voltage $V_{out} = V_A$ until $V_{in} = V_d$ and then falls to V_B. The output remains at V_B for $V_{in} > V_d$. Similarly, if V_{in} is initially $> V_d$ and gradually reduced, V_{out} remains at V_B until $V_{in} = V_c$, and then rises to V_A for all values $V_{in} < V_c$.

(i) Explain why the circuit behaves in this fashion, and

(ii) calculate the values of V_c and V_d.

42. A 440 volt DC shunt motor has an armature resistance of 0.5 *Ohm* and a field resistance of 220 *Ohms*. The motor takes 6 *amp* of current when idle on a 440 volt line. Calculate the efficiency of the motor at full load, when the current is 75 *amp*.

43. Assume that an analog voice signal which occupies a band from $300Hz$ to $3400Hz$, is to be transmitted over a Pulse Code Modulation (PCM) system. The signal is sampled at a rate of 8000 samples/sec. Each sample value is represented by 7 information bits plus 1 parity bit. Finally, the digital signal is passed through a raised cosine roll-off filter with the roll-off factor of 0.25. Determine
(i) whether the analog signal can be exactly recovered from the digital signal;
(ii) the bit duration and the bit rate of the PCM signal before filtering;
(iii) the bandwidth of the digital signal before and after filtering;
(iv) the signal to noise ratio at the receiver end (assume that the probability of bit error in the recovered PCM signal is zero).

44. A causal LTI discrete-time system develops an output

$$y[n] = (0.4)^n u[n] - (0.3)(0.4)^{n-1} u[n-1]$$

for an input $x[n] = (0.2)^n u[n]$.
(i)Determine the transfer function of the system and also the difference equation characterizing the system.
(ii) Develop a canonical direct form II realization of the system with no more than three multipliers OR a Parallel Form I realization of the system.
(iii) Determine the impulse response of the system.
(iv) Determine the output $y[n]$ of the system for an input $x[n] = (0.3)^n u[n] - (0.4)(0.3)^{n-1} u[n-1]$.

(v) COMPUTER SCIENCE

45. Let A be an $n \times n$ matrix such that for every 2×2 sub-matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ of A, if $a < b$ then $c \leq d$.

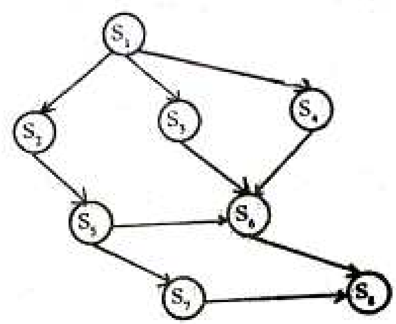
Note that for every pair of rows i and j, if a_{ik} and $a_{j\ell}$ are the largest elements in i-th and j-th rows of A, respectively, then $k \leq \ell$ (as illustrated in the 5×5 matrix below).

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 1 \\ 7 & 8 & 5 & 6 & 4 \\ 2 & 3 & 6 & 6 & 5 \\ 5 & 6 & 9 & 10 & 7 \\ 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

(i) Write an algorithm for finding the maximum element in each row of the matrix with time complexity $O(n\text{log}n)$.

(ii)Establish its correctness, and justify the time complexity of the proposed algorithm.

46. Consider the precedence graph shown in figure below.



Can this precedence graph be expressed using only concurrent statements? If so, how? If not, why? How can this precedence graph be expressed if semaphores are also used?

47. Let $S = \{x_1, x_2, ..., x_n\}$ be a set of n integers. A pair (x_i, x_j) is said to be the closest pair if $|x_i - x_j| \leq |x_{i'} - x_{j'}|$, for all possible pairs $(x_{i'}, x_{j'})$, $i', j' = 1, 2, ..., n, i' \neq j'$.

   (a) Describe a method for finding the closest pair among the set of integers in S using $O(n\ log_2 n)$ comparisons.

   (b) Now suggest an appropriate data structure for storing the elements in S such that if a new element is inserted to the set S or an already existing element is deleted from the set S, the current closest pair can be reported in $O(log_2 n)$ time.

   (c) Briefly explain the method of computing the current closest pair, and necessary modification of the data structure after each update. Justify the time complexity.

48. Consider a file consisting of 100 blocks. Assume that each disk I/O operation accesses a complete block of the disk at a time. How many disk I/O operations are involved with contiguous and linked allocation strategies, if one block is (i) added at the beginning? (ii) added at the middle? (iii) removed from the beginning? (iv) removed from the middle?

49. (a) Let M_1 be a deterministic finite automation which accepts a language $L_1 \subseteq \{0,1\}^*$. Define

$$L_2 = \{xb : b \in \{0,1\}, x \in L_1 \ \text{and} \ xb \in L_1\}.$$

Construct a deterministic finite automation M_2 which accepts L_2. Show that if M_1 has n states, it is sufficient for M_2 to have $2n$ states.

(b) Consider the grammar $G = (V,\ T,\ P,\ S)$, where $V = \{S,\ A_1,\ A_2,\ B_1,\ B_2\}$, $T = \{a,\ b\}$ and P consists of the productions given below. (Note that ϵ denotes the empty string.)

$$\begin{aligned} S &\rightarrow A_1B_1|A_2B_2 \\ A_1 &\rightarrow aA_1|\epsilon \\ A_2 &\rightarrow aaA_2|\epsilon \\ B_1 &\rightarrow bbB_1|b \\ B_2 &\rightarrow bbB_2|\epsilon \end{aligned}$$

Show that for each $i, j \geq 0$, the string a^ib^j has at most one leftmost derivation in G.

50. Consider a $100mbps$ token ring network with 10 stations having a ring latency of $50\mu s$ (the time taken by a token to make one complete rotation around the network when none of the stations is active). A station is allowed to transmit data when it receives the token, and it releases the token immediately after transmission. The maximum allowed holding time for a token (THT) is $200\mu s$.

(i) Express the maximum efficiency of this network when only a single station is active in the network.

(ii) Find an upper bound on the token rotation time when all stations are active.

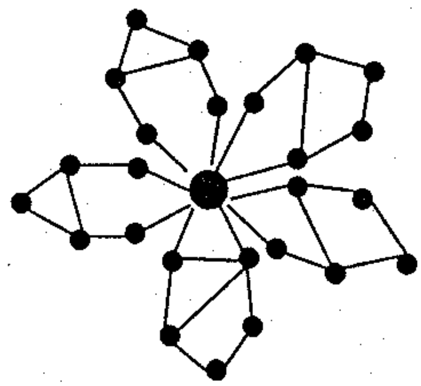
(iii) Calculate the maximum throughput rate that one host can achieve in the network.

51. An undirected graph $G = (V, E)$ with $kn + 1$ nodes is a k-daisy if it has a collection of k petals $p_1, p_2, \ldots, p_k$ $(p_i \subseteq V)$ such that
(i) $|p_i| = n + 1$

(ii) $\exists c \in V$ such that $p_i \cap p_j = \{c\}$ if $i \neq j$

(iii) $\forall i, \exists$ a simple cycle in G through all the vertices of p_i.

For example, see the following figure. Prove that the decision problem



of testing whether a given graph G is a 5-daisy, is NP-complete.

52. (a) A functional dependency $\alpha \to \beta$ is called a *partial* dependency if there is a proper subset γ of α such that $\gamma \to \beta$. Show that every partial dependency is a transitive dependency.

(b) Let $R = (A, B, C, D, E)$ be a schema with the set $F = \{A \to BC, CD \to E, B \to D, E \to A\}$ of functional dependencies. Suppose R is decomposed into two schema $R_1 = (A, B, C)$ and $R_2 = (A, D, E)$

(i) Is this decomposition loss-less? Justify.

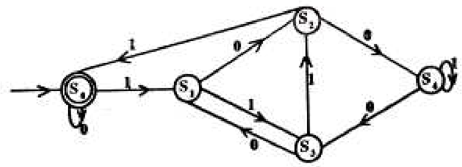
(ii) Is this decomposition dependency preserving? Justify.

(c) Consider the relations $r_1(A, B, C)$, $r_2(C, D, E)$ and $r_3(E, F)$. Assume that the set of all attributes constitutes the primary keys of these relations, rather than the individual ones. Let $V(C, r_1)$ be 500, $V(C, r_2)$ be 1000, $V(E, r_2)$ be 50, and $V(E, r_3)$ be 150, where $V(X, r)$ denotes the number of distinct values that appear in relation r for attribute X. If r_1 has 1000 tuples, r_2 has 1500 tuples, and r_3 has 750 tuples, then give the ordering of the natural join $r_1 \bowtie r_2 \bowtie r_3$ for its efficient computation. Justify your answer.

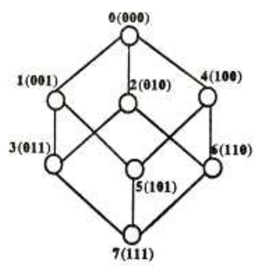
53. Assume that the following finite automaton shown in figure below, over $\sum = \{0, 1\}$ scans input strings from left to right.

(a) Describe the set of integers represented by the binary strings accepted by the above machine.

(b) Suggest a regular grammar generating the above language.



54. Let $G_n(V, E)$ be an undirected graph, such that $|V| = 2^n$, when n is a positive integer ≥ 2; the nodes are labelled as $0, 1, 2, \ldots.2^{n-1}$; and two nodes v_i and v_j are adjacent, i.e., $(v_i, v_j) \in E$, if their corresponding binary representations differ exactly in one bit position.
For example, G_3 is shown in figure below. Prove that
(i) $G_n(V, E)$ is a bipartite graph;
(ii) $G_n(V, E)$ admits a Hamiltonian cycle.



55. (a) What are the conditions which must be satisfied by a solution to the *critical section* problem?

(b) Consider the following solution to the critical section problem for two processes. The two processes, P_0 and P_1, share the following variables:

```
var flag : array [0..1] of Boolean;
                        (* initially false *)
    turn : 0..1;
```

The program below is for process P_i ($i = 0$ or 1) with process P_j ($j = 1$ or 0) being the other one.

```
repeat
  flag[i] <- true;
  while (flag[j])
    do if (turn = j)
```

```
            then begin
                   flag[i] <- false;
                   while (turn = j) do skip;
                 end;
      ...
```

CRITICAL SECTION

```
      ...
    turn <- j;
    flag[i] <- false;
     ...
```

REMAINDER SECTION

```
     ...
  until false;
```

Does this solution satisfy the required conditions?

(c) If a binary signal is sent over a 3 kHz channel whose signal-to-noise ratio is 20 db. What is the maximum achievable data-rate?

56. (a) Construct an AVL tree of height 5 with minimum number of nodes.

(b) Consider a B-tree of order 3.

(i) Trace the insertion of the keys a, g, f, b, k, d, h, m into an initially empty tree, in lexicographic order.

(ii) Sketch the B-tree upon deletion of keys h, d.