

Test Code : RC (Short Answer Type) 2010

JRF in Computer Science

The candidates for Junior Research Fellowship in Computer and Communication Sciences will have to take two tests - Test MIII (objective type) in the forenoon session and Test RC (short answer type) in the afternoon session. The RC test booklet will have two groups as follows:

GROUP A

A test for all the candidates in basics of computer programming.

GROUP B

A test, divided into five sections in the following areas at M.Sc., M.E., M.Tech. level:

- Mathematics,
- Statistics,
- Physics,
- Radiophysics/Telecommunication/Electronics/Electrical Engg., and
- Computer Science.

A candidate has to answer questions from **only** one of these sections in GROUP B, according to his/her choice.

The syllabi and sample questions of the RC test are given overleaf.

Syllabus

Elements of Computing:

Basics of programming (using pseudo-codes), Elementary data types and arrays.

Mathematics:

Graph theory and combinatorics: Graphs and digraphs, paths and cycles, trees, Eulerian graphs, Hamiltonian graphs, chromatic numbers, planar graphs, tournaments, inclusion-exclusion principle, pigeon-hole principle.

Linear algebra: Vector spaces, basis and dimension, linear transformations, matrices, rank, inverse, determinant, systems of linear equations, characteristic roots (eigen values) and characteristic vectors (eigen vectors), orthogonality and quadratic forms.

Abstract algebra: Groups, subgroups, cosets, Lagrange's theorem, normal subgroups and quotient groups, permutation groups, rings, subrings, ideals, integral domains, fields, characteristic of a field, polynomial rings, unique factorization domains, field extensions, finite fields.

Elementary number theory: Elementary number theory, divisibility, congruences, primality.

Calculus and real analysis: Real numbers, basic properties, convergence of sequences and series, limits, continuity, uniform continuity of functions, differentiability of functions of one or more variables and applications, indefinite integral, fundamental theorem of integral calculus, Riemann integration, improper integrals, double and multiple integrals and applications, sequences and series of functions, uniform convergence.

Statistics:

Probability theory and distributions: Basic probability theory, discrete and continuous distributions, moments, characteristic functions, Markov chains.

Estimation and inference: Sufficient statistics, unbiased estimation, maximum likelihood estimation, consistency of estimates, most powerful and uniformly most powerful tests, unbiased tests and uniformly most powerful unbiased tests, confidence sets, Bayesian methods.

Linear models: Gauss-Markov set up and least squares theory, multiple linear regression, one and two way analysis of variance.

Multivariate analysis: Multivariate normal distribution, Wishart distribution, Hotelling's T^2 test, principal component analysis, multiple and canonical correlations, discriminant analysis, cluster analysis, factor analysis.

Physics:

Classical mechanics: Newtonian mechanics, study of system of particles, Lagrangian and Hamiltonian formalisms, symmetries and conservation laws, motion in central field of force, collisions and scattering, small oscillations and normal modes, wave motion, special theory of relativity.

Electrodynamics: Electrostatics, boundary conditions, method of images, multipole expansion, magnetostatics, electric fields and magnetic fields in matter, electromagnetic induction, Maxwell's equations, conservation laws, gauge invariance, electromagnetic waves, potentials and fields, electrodynamics of a charged particle in electric and magnetic fields, radiation, electrodynamics and relativity.

Thermodynamics and statistical physics: Laws of thermodynamics and its consequences, statistical basis of thermodynamics, thermodynamic potentials and Maxwell's relations, density matrix formulation, ensembles, partition function and its uses, classical and quantum statistics, simple gas problems, black body radiation and Planck's distribution law, Bose-Einstein condensation, Einstein and Debye models.

Quantum physics and applications: Basics of quantum mechanics, two-body problem, central potential, angular momentum algebra, perturbation theory, Fermi's golden rule, quantum particles in electromagnetic fields, matrix mechanics, spin-statistics connection, approximation methods: stationary and time dependent problems.

Condensed matter physics: Crystal structures, 2D, 3D, reciprocal lattices, lattice vibrations, diffraction and structure factor, specific heat of solids, electron motion in periodic potential, electronic band structures, doped semiconductors, density of states, transport properties, polarization, Clausius-Mossotti equation; piezo-, pyro- and ferro-electricity; dia-, para-, ferro-, antiferro-magnetism, basics of superconductivity.

Atomic and molecular physics: Energy spectrum of an electron, hydrogen atom, electron spin, LS and JJ coupling, relativistic correction, selection rules, hyperfine structure, Zeeman, Paschen-back and Stark effect.

Nuclear and particle physics: Basic nuclear properties, nuclear force, nuclear models, radioactive decays, elementary particle physics.

Electronics: Basics of semiconductor physics, operational amplifier and its applications, LCR circuits, digital integrated circuits, basic principles and applications of A/D and D/A converters.

Vibrations and waves: Forced vibrations, coupled vibrations, strings, small oscillations.

Radiophysics/Telecommunication/Electronics/Electrical Engg.:

Digital circuits and systems: Boolean algebra, gates and logic circuits, combinational and sequential circuits, A/D and D/A converters.

Circuit theory: Kirchoff's Laws, theorem of superposition, Thevenin's Theorem, Norton's Theorem, single-phase and three-phase A.C. circuits.

Linear electronic devices and circuits: Transistors, amplifiers including feedback, oscillators, operational amplifiers.

Semiconductor physics: Energy bands, junction devices, transport theory.

Digital communication: Modulation theory, information theory, coding and channel capacity, digital transmission.

Digital signal processing: Sampling, linear time invariant systems, Z-transform, discrete-time Fourier and discrete Fourier transforms, discrete-time random signals, filter structures, basics of multirate processing.

Linear control theory: Continuous and discrete linear systems, state-variable representation, differential and difference equations, stability, controllability, observability.

Electrical machines: DC motors and generators, transformers, induction motors, measuring instruments (voltmeter, ammeter and galvanometer).

Computer Science:

Discrete Mathematics: Elementary counting techniques, principles of inclusion-exclusion, recurrence relations, generating functions, propositional logic.

Programming languages: Fundamental concepts - abstract data types, procedure call and parameter passing, C language.

Data structures: Stack, queue, linked list, binary tree, heap, AVL tree, B-tree.

Design and analysis of algorithms: Order notation, sorting, selection, searching, hashing, string handling algorithms, graph algorithms, NP-completeness.

Computer organization and architecture: Number representation, computer arithmetic, memory organization, I/O organization, microprogramming, pipelining, instruction level parallelism.

Operating systems: Memory management, processor management, critical section, deadlocks, device management, concurrency control.

Formal languages and automata theory: Finite automata and regular expression, context-free grammars, Turing machines, elements of undecidability.

Principles of compiler construction: Lexical analyzer, symbol table, parser, code optimization.

Database systems: Relational model, relational algebra, relational calculus, functional dependency, normalization (including multi-valued dependencies), query processing and optimization.

Computer networks: Layered network structures, network security, LAN technology - Bus/tree, Ring, Star; data communications - data encoding, flow control, error detection/correction, TCP/IP networking including IPv4.

Sample Questions

*Note that all questions in the sample set are not of equal difficulty.
They may not carry equal marks in the test.*

GROUP A

ELEMENTS OF COMPUTING

A1. Show how two complex numbers $(a+ib)$ and $(c+id)$ may be multiplied using only three multiplications of real numbers, where $i = \sqrt{-1}$. You may use any number of additions and subtractions.

A2. Consider the pseudo-code given below.

Input: Integers b and c .

1. $a_0 \leftarrow \max(b, c)$, $a_1 \leftarrow \min(b, c)$.
2. $i \leftarrow 1$.
3. Divide a_{i-1} by a_i . Let q_i be the quotient and r_i the remainder.
4. If $r_i = 0$ then go to Step 8.
5. $a_{i+1} \leftarrow a_{i-1} - q_i * a_i$.
6. $i \leftarrow i + 1$.
7. Go to Step 3.
8. Print a_i .

What is the output of the above algorithm? What is the mathematical relation between the output a_i and the two inputs b and c ?

A3. Write the output of the following pseudo-code:

1. for ($n = 15$, downto 2, step -2)
2. if ($n > 10$)
3. then $n \leftarrow n + 1$ and print n ;
4. else $n \leftarrow n - 1$ and print n ;
5. endfor

A4. Consider the sequence $a_n = a_{n-1} a_{n-2} + n$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 1$. Is a_{2006} odd? Justify your answer.

A5. Given an array of n integers, write a pseudo-code for reversing the contents of the array without using another array. For example, for the array
 10 15 3 30 3
 the output should be
 3 30 3 15 10.
 You may use one temporary variable.

A6. Derive an expression for the maximum number of regions that can be formed within a circle by drawing n chords.

A7. A function *PrintRec* is defined as follows:

```

PrintRec(A, B, C, n)
  begin
    if  $n > 0$ 
      begin
        print B;
        PrintRec(B, A, C,  $n - 1$ );
        print A;
        PrintRec(C, A, B,  $n - 2$ );
      end
    end
  end

```

Find the output for the function call *PrintRec*($x, y, z, 3$). Show the intermediate steps of your derivation.

A8. Given $A = \{1, 2, 3, \dots, 70\}$, show that for any six elements a_1, a_2, a_3, a_4, a_5 and a_6 belonging to A , there exists one pair a_i and a_j for which $|a_i - a_j| \leq 14$.

A9. Let $p(x)$ and $q(x)$ be two real polynomials of degrees m and n respectively. To compute the product $p(x) \times q(x)$, how many operations (i.e. additions and multiplications of reals) do you need to perform? Write down an algorithm to find the product of two polynomials.

GROUP B

(i) MATHEMATICS

M1. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function for which there does not exist any $x \in [0, 1]$ such that both $f(x) = 0$ and $f'(x) = 0$. Show that f has only a finite number of zeros in $[0, 1]$.

[$f'(x)$ denotes the derivative of f at x].

(b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f'(x)$ exists and is continuous in $[0, 1]$. Show that

$$\lim_{x \downarrow 0} \frac{1}{x^2} \int_0^x (x - 3y)f(y)dy = -\frac{f(0)}{2}.$$

[$x \downarrow 0$ denotes: x decreases to zero.]

M2. (a) Let $f : [0, 1] \rightarrow [0, 1]$ be such that

$$f(x) = nx - [nx]; \quad \frac{1}{n} < x \leq \frac{1}{n-1}; \quad n = 2, 3, 4, \dots; \quad x \neq 0$$

and $f(0) = 0$. Show that $\int_0^1 f(x)dx$ exists and find its value.

Note: $[y] =$ Largest integer $\leq y$; $y \in \mathbb{R}$.

(b) Let $f : [0, 1] \rightarrow (0, \infty)$ be continuous. Let

$$a_n = \left(\int_0^1 (f(x))^n dx \right)^{\frac{1}{n}}; \quad n = 1, 2, 3, \dots$$

Find $\lim_{n \rightarrow \infty} a_n$.

M3. (a) Suppose f is a continuous real valued function on $[0, 1]$. Show that

$$\int_0^1 xf(x) dx = \frac{1}{2}f(\xi)$$

for some $\xi \in [0, 1]$.

(b) For every $x \geq 0$, prove that

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}},$$

for a unique $\theta(x)$, $0 < \theta(x) < 1$. Also prove that,

(i) $\frac{1}{4} \leq \theta(x) \leq \frac{1}{2}$, and

(ii) $\lim_{x \rightarrow 0} \theta(x) = \frac{1}{4}$ and $\lim_{x \rightarrow \infty} \theta(x) = \frac{1}{2}$.

- M4. (a) Show that $f(x) = e^{|x|} - x^5 - x - 2$ has at least two real roots, where e is the base of natural logarithms.
- (b) Let $\sum a_n$ be a convergent series such that $a_n \geq 0$ for all n . Show that $\sum \sqrt{a_n}/n^p$ converges for every $p > \frac{1}{2}$.

- M5. (a) Let G be the set of all non-singular 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where the elements a, b, c, d belong to the field of order 3. Using matrix multiplication as the operation in G what is the order of group G ? Is G abelian? Justify your answers.
- (b) Prove that $\text{Aut}(Q, +) \simeq Z_2$, where Z_2 is the group consisting of only two elements; and $\text{Aut}(Q, +)$ is the automorphism group of $(Q, +)$.

- M6. Let $R = (S, +, \cdot, 0, 1)$ be a commutative ring and n be a positive integer such that $n = 2^k$ for some positive integer k .

- (a) Show that for every $a \in S$

$$\sum_{i=0}^{n-1} a^i = \prod_{i=0}^{k-1} (1 + a^{2^i}).$$

- (b) Let $m = w^{\frac{n}{2}} + 1$ where $w \in S$, $w \neq 0$. Show that for $1 \leq p < n$,

$$\sum_{i=0}^{n-1} w^{ip} \equiv 0 \pmod{m}.$$

- M7. (a) Show that there is a basis consisting of only symmetric and skew-symmetric matrices for the vector space of all $n \times n$ real matrices over \mathbb{R} .
- (b) Does there exist a linear transformation from \mathbb{R}^3 to \mathbb{R}^3 such that $(0,1,1)$, $(1,1,0)$, $(1,2,1)$ are mapped to $(1,0,0)$, $(0,1,0)$, $(0,1,1)$ respectively? Justify your answer.
- M8. (a) Let A be a square matrix such that $A^2 = A$. Show that all eigenvalues of A are 0 or 1.
- (b) Let A be a symmetric matrix whose eigenvalues are 0 or 1. Show that $A^2 = A$.

(c) Suppose A is an $n \times n$ matrix such that $A^2 = A$. If $\text{rank}(A) = r$ and $\text{rank}(I - A) = s$, show that $r + s = n$.

M9. Let k be a positive integer. Let $G = (V, E)$ be the graph where V is the set of all strings of 0's and 1's of length k , and $E = \{(x, y) : x, y \in V, x \text{ and } y \text{ differ in exactly one place}\}$.

- (a) Determine the number of edges in G .
- (b) Prove that G has no odd cycle.
- (c) Prove that G has a perfect matching.
- (d) Determine the maximum size of an independent set in G .

M10. Let T be a tree with n vertices. For vertices u, v of T , define $d(u, v)$ to be the number of edges in a path from u to v . Let $W(T)$ be the sum of $d(u, v)$ over all $\binom{n}{2}$ pairs of vertices $\{u, v\}$.

(a) Suppose the tree T is a path on n vertices. Show that

$$W(T) = \frac{1}{2} \sum_{k=1}^{n-1} (k^2 + k).$$

- (b) Now, suppose T is a star graph on n vertices. Show that $W(T) = (n-1)^2$. (N.B. The edge set of the star graph is equal to $\{(u_1, u_i) : 2 \leq i \leq n\}$.)
- (c) Hence or otherwise show that for any tree T ,

$$(n-1)^2 \leq W(T) \leq \frac{1}{2} \sum_{k=1}^{n-1} (k^2 + k).$$

M11. (a) Show that, given $2^n + 1$ points with integer coordinates in \mathbb{R}^n , there exists a pair of points among them such that all the coordinates of the midpoint of the line segment joining them are integers.

(b) Find the number of functions from the set $\{1, 2, 3, 4, 5\}$ onto the set $\{1, 2, 3\}$.

M12. (a) A set S contains integers 1 and 2, and all integers of the form $3x + y$ where x and y are distinct elements of S . What is S ? Justify your answer.

(b) Let $\phi(n)$ denote the number of positive integers m relatively prime to n ; $m < n$. Let $n = pq$ where p and q are prime numbers. Then show that $\phi(n) = (p-1)(q-1) = pq(1 - \frac{1}{q})(1 - \frac{1}{p})$.

M13. Consider the $n \times n$ matrix $A = ((a_{ij}))$ with $a_{ij} = 1$ for $i < j$ and $a_{ij} = 0$ for $i \geq j$. Let

$$V = \{f(A) : f \text{ is a polynomial with real coefficients}\}.$$

Note that V is a vector space with usual operations. Find the dimension of V , when (a) $n = 3$, and (b) $n = 4$. Justify your answer.

M14. Consider the following system of equations over a field \mathbf{F} .

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2, \end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbf{F}$. State the conditions for which the above system of equations has (a) no solution, (b) a unique solution and (c) more than one solution.

Next consider the following equations over the field \mathbf{Z}_p , the field of integers modulo p , where p is a prime.

$$\begin{aligned} 7x - y &= 1 \\ 11x + 7y &= 3 \end{aligned}$$

For what value of p does this system of equations have (a) no solution, (b) a unique solution and (c) more than one solution? Find the solution in the case of (b).

M15. In each of the following four problems, two sets S and T are given. You need to define a continuous and onto function $f : S \rightarrow T$ in each of the cases. You need to provide mathematical reasons if you cannot define such an f .

- (a) $S = (0, 1)$, $T = (0, 1]$.
- (b) $S = (0, 1]$, $T = (0, 1)$.
- (c) $S = (0, 1) \cup (2, 3)$, $T = (0, 1)$.
- (d) $S = (0, 1)$, $T = (0, 1) \cup (2, 3)$.

- M16. You are given 49 balls of colour red, black and white. It is known that, for any 5 balls of the same colour, there exist at least two among them possessing the same weight. The 49 balls are distributed in two boxes. Prove that there are at least 3 balls which lie in the same box possessing the same colour and having the same weight.
- M17. Consider a sequence $\{a_n\}$ such that $0 < a_1 < a_2$ and $a_i = \sqrt{a_{i-1}a_{i-2}}$ for $i \geq 3$.
- (a) Show that $\{a_{2n-1} : n = 1, 2, \dots\}$ is an increasing sequence and $\{a_{2n} : n = 1, 2, \dots\}$ is a decreasing sequence.
 - (b) Show that $\lim_{n \rightarrow \infty} a_n$ exists.
 - (c) Find $\lim_{n \rightarrow \infty} a_n$.
- M18. If every vertex of a graph $G = (V, E)$ has degree at least $\frac{|V|}{2}$, then prove that there is a simple cycle containing all the vertices. Note that, a simple cycle with a vertex set $U \subseteq V$ is a cycle where each vertex of U appears only once if one traverses along the cycle.

(ii) STATISTICS

- S1. (a) Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables satisfying $X_{n+1} = X_n + Z_n$ (addition is modulo 5), where $\{Z_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables with common distribution $P(Z_n = 0) = 1/2, P(Z_n = -1) = P(Z_n = +1) = 1/4$. Assume that X_1 is a constant belonging to $\{0, 1, 2, 3, 4\}$. What happens to the distribution of X_n as $n \rightarrow \infty$?
- (b) Let $\{Y_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables with a common uniform distribution on $\{1, 2, \dots, m\}$. Define a sequence of random variables $\{X_n\}_{n \geq 1}$ as $X_{n+1} = \text{MAX}\{X_n, Y_n\}$ where X_1 is a constant belonging to $\{1, 2, \dots, m\}$. Show that $\{X_n\}_{n \geq 1}$ is a Markov chain and classify its states.
- S2. Let x_1, x_2, \dots, x_n be a random sample of size n from the gamma distribution with density function

$$f(x, \theta) = \frac{\theta^k}{\Gamma(k)} e^{-\theta x} x^{k-1}, \quad 0 < x < \infty,$$

where $\theta > 0$ is unknown and $k > 0$ is known. Find a minimum variance unbiased estimator for $\frac{1}{\theta}$.

- S3. Let $0 < p < 1$ and $b > 0$. Toss a coin once where the probability of occurrence of head is p . If head appears, then n independent and identically distributed observations are generated from Uniform $(0, b)$ distribution. If the outcome is tail, then n independent and identically distributed observations are generated from Uniform $(2b, 3b)$ distribution. Suppose you are given these n observations X_1, \dots, X_n , but not the outcome of the toss. Find the maximum likelihood estimator of b based on X_1, \dots, X_n . What happens to the estimator as n goes to ∞ ?
- S4. Let X_1, X_2, \dots be independent and identically distributed random variables with common density function f . Define the random variable N as
- $$N = n, \text{ if } X_1 \geq X_2 \geq \dots \geq X_{n-1} < X_n; \text{ for } n = 2, 3, 4, \dots$$
- Find $\text{Prob}(N = n)$. Find the mean and variance of N .

- S5. (a) Let X and Y be two random variables such that

$$\begin{pmatrix} \log X \\ \log Y \end{pmatrix} \sim N(\mu, \Sigma).$$

Find a formula for $\phi(t, r) = E(X^t Y^r)$, where t and r are real numbers, and E denotes the expectation.

- (b) Consider the linear model $y_{n \times 1} = A_{n \times p} \beta_{p \times 1} + \epsilon_{n \times 1}$ and the usual Gauss-Markov set up where $E(\epsilon) = 0$ and $D(\epsilon) = \sigma^2 I_{n \times n}$, E denotes the *expectation* and D denotes the covariance matrix. Assume that A has full rank. Show that $Var(\beta_1^{LS}) = (\alpha - \Gamma^T B^{-1} \Gamma)^{-1} \sigma^2$ where

$$A^T A = \begin{bmatrix} \alpha_{1 \times 1} & \Gamma_{1 \times p-1}^T \\ \Gamma_{1 \times p-1} & B_{p-1 \times p-1} \end{bmatrix}$$

and β_1^{LS} = the least square estimate of β_1 , the first component of the vector β , Var denotes the variance, and A^T denotes the transpose of the matrix A .

- S6. Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two independent and identically distributed multivariate random vectors with mean $\mathbf{0}$ and covariance matrix $\sigma^2 \mathbf{I}_n$, where $\sigma^2 > 0$ and \mathbf{I}_n is the $n \times n$ identity matrix.

- (a) Show that $\mathbf{X}^T \mathbf{Y} / (\|\mathbf{X}\| \cdot \|\mathbf{Y}\|)$ and $V = \sum (X_i^2 + Y_i^2)$ are independent.

(Here, $\|(a_1, \dots, a_n)\| = \sqrt{a_1^2 + \dots + a_n^2}$).

- (b) Obtain the probability density function of $(\sum_{i=1}^n X_i^2 / \sum_{i=1}^n Y_i^2)$.

- S7. Let X_1, X_2, \dots, X_n be independent random variables. Let $E(X_j) = j\theta$ and $V(X_j) = j^3 \sigma^2$, $j = 1, 2, \dots, n$, $-\infty < \theta < \infty$ and $\sigma^2 > 0$. Here $E(X)$ denotes the expectation and $V(X)$ denotes the variance of the random variable X . It is assumed that θ and σ^2 are unknown.

- (a) Find the best linear unbiased estimator for θ .
 (b) Find the uniformly minimum variance unbiased estimate for θ under the assumption that X_i 's are normally distributed; $1 \leq i \leq n$.

- S8. Let (X, Y) follow the bivariate normal distribution. Let *mean* of X = *mean* of Y = 0, variance of X = variance of Y = 1, and the correlation coefficient between X and Y be ρ . Find the correlation coefficient between X^3 and Y^3 .

S9. Let the probability density function of a random variable X be $f(x)$, $-\infty < x < \infty$, and we have two hypotheses

$$H_0 : f(x) = (2\pi)^{-1/2} \exp(-x^2/2) \text{ and}$$

$$H_1 : f(x) = (1/2) \exp(-|x|).$$

In connection with testing H_0 against H_1 , derive the most powerful test at level $\alpha = 0.05$.

S10. Let X_1, X_2, \dots, X_n be independent and identically distributed observations with a common exponential distribution with mean μ . Show that there is no uniformly most powerful test for testing $H_0 : \mu = 1$ against $H_A : \mu \neq 1$ at a given level $0 < \alpha < 1$, but there exists a uniformly most powerful unbiased test. Derive that test.

S11. (a) An unbiased die is rolled once. Let the score be $N \in \{1, 2, \dots, 6\}$. The die is then rolled n times. Let X be the maximum of these n scores. Find the probability of the event $(X = 6)$.

(b) The unit interval $(0,1)$ is divided into two sub-intervals by picking a point at random from the interval. Let Y and Z be the lengths of the longer and shorter sub-intervals, respectively. Find the distribution of Z and show that $\frac{Y}{Z}$ does not have a finite expectation.

S12. Let X_1, X_2, X_3 be independent and identically distributed observations with a common double exponential distribution with density

$$f(x, \theta) = \frac{1}{2} \exp(-|x - \theta|), \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Suppose that the observations are all distinct.

(a) Find the maximum likelihood estimator of θ . Give a complete argument for your answer.

(b) Suppose it is known that $-10 \leq \theta \leq 10$. Find the maximum likelihood estimator of θ . Justify your answer.

S13. Consider the following linear model

$$y_{ij} = \alpha_i + \beta_j + e_{ij}, \quad i = 1, 2; \quad j = 1, 2, 3;$$

- (a) What is the rank of the error-space? Justify your answer.
- (b) Write down any linear function of observations that belongs to
(i) estimation space, (ii) error space.
- (c) Write down a parametric function that is not estimable. Justify your answer.
- S14. Let $A = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ be the space obtained by tossing a coin three times. Let $f : A \rightarrow (0, \infty)$ and $x_1 \in A$. For any $x_i \in A$, x_{i+1} is found in the following way. Toss a fair coin three times and let the outcome be z . If $f(z) \geq f(x_i)$ then $x_{i+1} = z$, otherwise $x_{i+1} = x_i$. What can you say about $\lim_{x \rightarrow 0} f(x_i)$? Justify your answer.
- S15. Let there be two classes C_1 and C_2 . Let the density function for class C_i be $p_i(x)$ for $i = 1, 2$ where $p_i(x) = ie^{-ix}$; $x > 0$, $i = 1, 2$. Let the prior probability for C_1 be 0.4 and the prior probability for C_2 be 0.6. Find the decision rule for classification of an observation, which provides the minimum probability of misclassification and find its value for that decision rule.
- S16. (a) Let $\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots, \underline{x}_{10}$ be a random sample of 10 observations from 100 dimensional multivariate normal distribution with dispersion matrix $\Sigma = ((\sigma_{ij}))_{100 \times 100}$. Let $\hat{\sigma}_{ij}$ be maximum likelihood estimate of σ_{ij} based on $\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots, \underline{x}_{10}$. Let $\hat{\Sigma} = ((\hat{\sigma}_{ij}))_{100 \times 100}$. Show that $rank(\hat{\Sigma}) \leq 10$.
- (b) Show that $X_1 + 2X_2$ is not sufficient for μ where X_1 and X_2 are two independent and identically distributed observations from $N(\mu, 1)$ distribution.
- S17. (a) Suppose $X \sim N(\theta, 1)$, where $\theta \geq 0$. For any fixed $x > 0$, define

$$g(\theta) = P_\theta(X^2 > x), \quad \theta \geq 0.$$

Show that $g(\theta)$ is a strictly increasing function of θ on $[0, \infty)$.

- (b) Suppose $\mathbf{X} \sim N_p(\boldsymbol{\theta}, \mathbf{I}_p)$, where $\boldsymbol{\theta} \in \mathbb{R}^p$, and \mathbf{I}_p is the identity matrix of order p . For any fixed $x > 0$, define

$$h(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(\mathbf{X}^T \mathbf{X} > x), \quad \boldsymbol{\theta} \in \mathbb{R}^p.$$

Assuming that $h(\boldsymbol{\theta})$ depends on $\boldsymbol{\theta}$ only through $\boldsymbol{\theta}^T \boldsymbol{\theta}$, show that $h(\boldsymbol{\theta})$ is a strictly increasing function of $\boldsymbol{\theta}^T \boldsymbol{\theta}$.

- S18. (a) Suppose independent observations are generated from the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \dots, n$, where x_1, \dots, x_n are known positive numbers, and $E(\epsilon_i) = 0 \forall i, \text{Var}(\epsilon_i) = \sigma^2 \forall i$. However, we fit the model $y_i = \beta_1 x_i + \epsilon_i$. Show that the bias in the least squares estimate of β_1 is at most β_0/\bar{x} .
- (b) Consider the linear model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}_n)$, where \mathbf{X} is an $n \times p$ matrix with entries 0, 1 and -1 . Assume that \mathbf{X} is of full column rank. Show that if $\hat{\beta}_j$ denotes the best linear unbiased estimate (BLUE) of β_j , then

$$\text{Var}(\hat{\beta}_j) \geq \sigma^2/n.$$

- S19. Suppose that \mathbf{X} has uniform distribution over the square defined by the four vertices $(0, \theta), (\theta, 0), (0, -\theta)$ and $(-\theta, 0)$ in the plane, where $\theta > 0$. Let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ be independent and identically distributed (i.i.d.) observations from this distribution. Find the maximum likelihood estimate of θ based on these observations.

- S20. Let X_1, \dots, X_n be a random sample from a Bernoulli(p) distribution. We want to test $H_0 : p = 0.4$ against $H_1 : p = 0.6$.

- (a) Find the most powerful test for testing H_0 against H_1 .
- (b) Use the Central Limit Theorem to determine the sample size needed so that the two probabilities of error of the test obtained in (a) are both approximately equal to α . Find the result in terms of z_α , the $(1 - \alpha)$ -th quantile of $N(0, 1)$.

(iii) PHYSICS

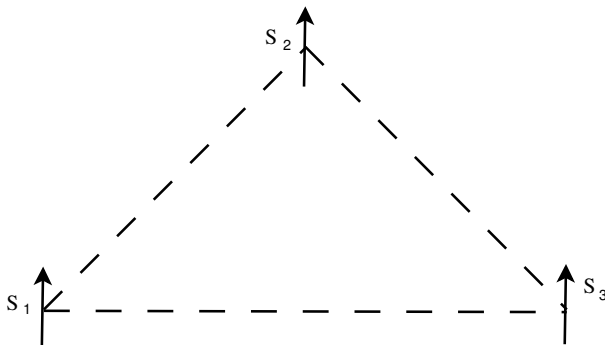
- P1. (a) In the electron spin-orbit interaction the two possible values of j are $l + \frac{1}{2}$ and $l - \frac{1}{2}$. Show that the expectation values of S_Z in the states $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ are $+\frac{m_j \hbar}{2l+1}$ and $-\frac{m_j \hbar}{2l+1}$ respectively. [All the symbols have their usual meanings.]
- (b) A particle in the infinite square well has the initial wave function

$$\Psi(x, 0) = A \sin^3 \frac{\pi x}{a}, (0 \leq x \leq a)$$

- (i) Determine A .
- (ii) Find $\Psi(x, t)$.
- (iii) Calculate $\langle x \rangle$ as a function of time.
- (c) Assume that three spins S_1, S_2, S_3 are arranged in the form of an equilateral triangle with each spin interacting with its two neighbors (see figure below). Each spin can take values $+1$ or -1 . The energy of this system in a magnetic field perpendicular to the plane of the triangle, is

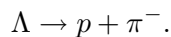
$$H = -J(S_1 S_2 + S_2 S_3 + S_3 S_1) - F(S_1 + S_2 + S_3).$$

Here J and F are constant parameters.



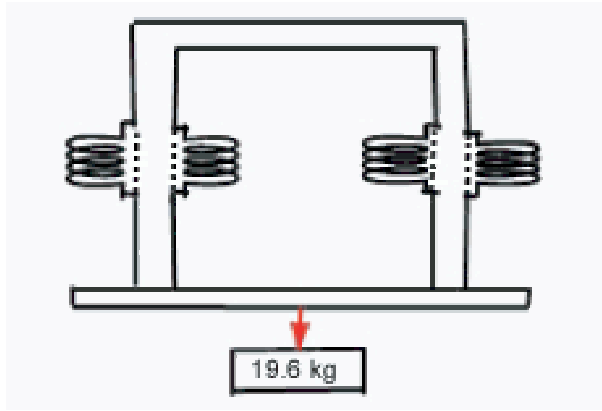
- (i) Find the partition function of the system.
- (ii) Find the average spin.

- P2. The Λ -particle is an unstable sub-atomic particle that can decay spontaneously into a proton and a negatively charged pion



In a certain experiment, the outgoing proton and pion were observed, both travelling in the same direction along the positive x -axis with momenta P_p and P_{π^-} respectively. Find the rest mass of Λ -particle given the rest masses of proton and pion to be m_p and m_{π^-} respectively.

- P3. A horse-shoe magnet is formed out of a bar of wrought iron of 50 cm length having cross section 6.28 cm^2 . An exciting coil of 500 turns is placed on each limb and connected in series. Find the exciting current necessary for the magnet to lift a load of 19.6 kg (see the figure given below) assuming that the load has negligible reluctance and makes close contact with the magnet. Relative permeability of iron is 700.



- P4. (a) A particle of mass m and charge e enters a homogeneous retarding electric field with a velocity v parallel to the direction of the field. How much time will the particle take to return to its initial position?
 (b) Find the electrostatic energy of a uniformly charged spherical shell of total charge q , and radii R_1 and R_2 ($R_2 > R_1$).
 (c) From the set of Maxwell's equations for electrodynamics, derive the charge conservation equation.

- P5. From the first law of thermodynamics deduce,

$$dQ = C_V dT + \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right] dV,$$

where the symbols have their usual meanings.

- P6. (a) Sketch the Fermi-Dirac distributions at zero temperature and at a small non-zero temperature.
- (b) Derive the relation between pressure and volume of a free non-relativistic electron gas at zero temperature.
- (c) Show that the heat capacity of a one dimensional monatomic lattice at temperature T is proportional to $\frac{T}{\theta}$ for low temperature in the Debye approximation. Here θ denotes the effective Debye temperature and $T \ll \theta$.

- P7. (a) Consider an eigenstate of a two-particle system, in one dimension, represented by the wave function

$$\psi(x_1, x_2) = e^{iP(x_1+x_2)/(2\hbar)} e^{-(Mk/2)^{\frac{1}{2}}(x_1-x_2)^2/(2\hbar)}.$$

Here x_1 and x_2 are the positions of the two particles of equal mass (M) moving in one dimension and interacting with a harmonic oscillator force $F = -k(x_1 - x_2)$.

- (i) Calculate the total energy associated with the relative motion.
- (ii) Also calculate the mean absolute value of the relative momentum.
- (b) A particle of mass m_1 MeV/ c^2 and kinetic energy k_1 MeV collides with a stationary particle of mass m_2 MeV/ c^2 . After the collision, the two particles stick together. Calculate
- (i) the initial momentum of the two-particle system, and
- (ii) the final velocity of the two-particle system.

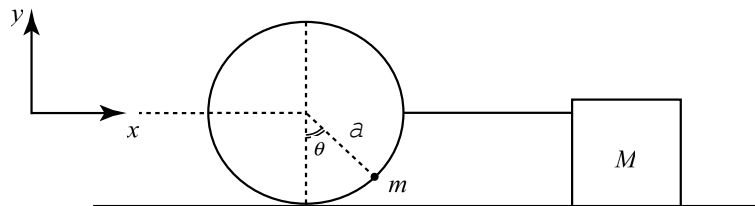
- P8. (a) A particle of mass m and kinetic energy $E > 0$ approaches an abrupt potential drop $V(x)$ given by

$$V(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ -V_0 & \text{if } x > 0. \end{cases}$$

Calculate the reflection coefficient if $E = \frac{V_0}{3}$.

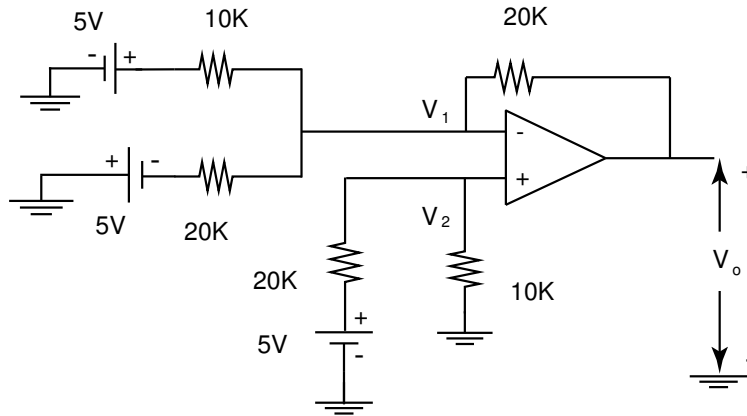
- (b) Consider a sequence of $2N$ ions of alternating charges $\pm q$ arranged on a line with a repulsive potential $\frac{A}{R^n}$ between nearest neighbours in addition to the usual Coulomb potential. Find the equilibrium separation R_0 and the equilibrium energy. Also evaluate the nearest neighbour distance when the potential energy is zero. (Neglect the surface effect).

- P9. (a) Two point charges of magnitude e are located at the end points of a line of length $2l$ in the xy plane with its midpoint passing through the origin. The line is rotating about the z -axis in the anti-clockwise direction with a constant angular velocity ω . Calculate the following:
- electric dipole moment of the system.
 - magnetic dipole moment of the system.
- (b) A parallel plate capacitor (having perfectly conducting plates) with plate separation d is filled with layers of two different materials. The first layer has dielectric constant ϵ_1 and conductivity σ_1 ; the second layer has dielectric constant ϵ_2 and conductivity σ_2 . Their thicknesses are d_1 and d_2 , respectively. A potential difference of V is applied across the capacitor. Neglecting the edge effect,
- calculate the electric field in each of the two dielectric media.
 - what is the current flowing through the capacitor?
 - what is the total surface charge density on the interface between the two layers?
- P10. A block of mass M is rigidly connected to a massless circular track of radius a fixed in a vertical plane on a horizontal table as shown in the figure. A particle of mass m is confined to move without friction on the circular track (in the vertical plane).

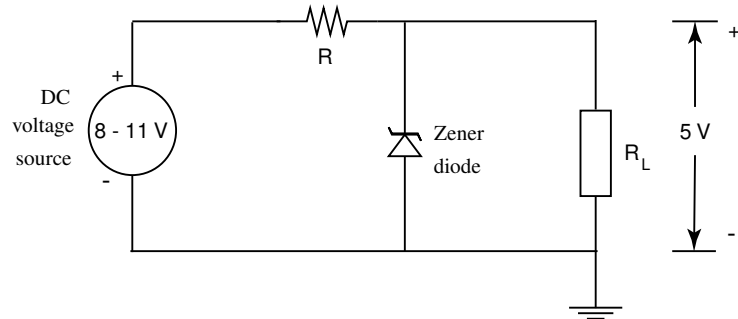


- Set up the Lagrangian using x and θ as the coordinates.
- Find the equations of motion.
- Solve the equation of motion for θ (for small θ).

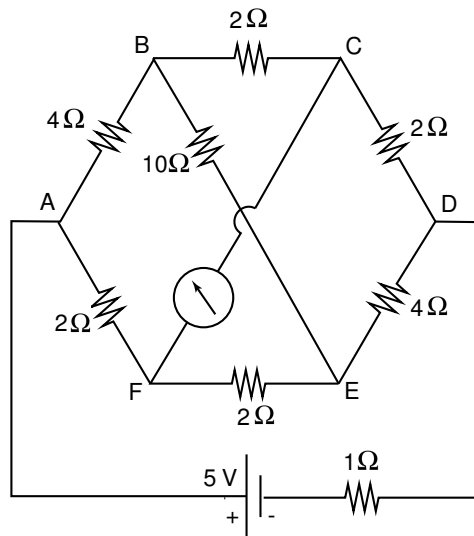
- P11. For an intrinsic semiconductor with a gap width of 1 eV, calculate the position of Fermi level at $T = 0^0 K$ and $T = 300^0 K$, if $m_h = m_e$, where m_e and m_h are effective masses of an electron and a hole respectively. Also calculate the density of free electrons and holes at $T = 300^0 K$ and $T = 600^0 K$, given that $\log_{10} e = 0.40$, $\left(\frac{2\pi\sqrt{m_e m_h} kT}{h^2}\right)^{\frac{3}{2}} = 0.5 \times 10^{19}/cc$. If the above semiconductor is now doped with a group V element with a doping concentration of $10^{14}/cc$, then compute the electron and hole densities of the doped semiconductor specimen.
- P12. (a) A negative feedback amplifier has a voltage gain of 100. Variations of the voltage gain up to $\pm 2\%$ can be tolerated for some specific application. If the open-loop gain variations of $\pm 10\%$ are expected owing to production spreads in device characteristics, determine the minimum value of the feedback ratio β and also the open loop gain to satisfy the above specification.
- (b) Calculate the output voltage V_0 for the following network:



- P13. (a) Consider the following circuit for deriving a +5 volt power supply to a resistive load R_L from an input d-c voltage source whose voltage may vary from 8V to 11V. The load R_L may draw a maximum power of 250 mW. The Zener diode has a breakdown voltage of 5 volts. Compute the maximum value of the resistance R and also the power dissipation requirements for R and the Zener diode. Assume that the minimum breakdown current of the Zener diode is negligible compared to the load current.



- (b) Consider the following circuit. Calculate the potential difference between the points F and C , as shown by the ideal voltmeter.



- P14. Consider a particle of charge e and mass m in constant crossed electric field \vec{E} and magnetic field \vec{B} given by

$$\vec{E} = (0, 0, E), \quad \vec{B} = (0, 0, B),$$

in coordinates $\vec{r} = (x, y, z)$.

- (a) Write the Schroedinger equation in a convenient gauge.
 (b) Separate variables and reduce it to a one-dimensional problem.

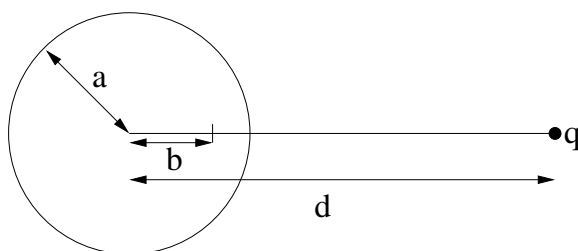
- (c) Calculate the expectation value of the velocity in the x -direction (sometimes called the drift velocity) in any energy eigenstate.

- P15. (a) Suppose the scalar potential $V = 0$ and the vector potential

$$\mathbf{A} = A_0 \sin(kx - \omega t) \hat{y},$$

where A_0 , ω and k are constants. Find the electric field \mathbf{E} and the magnetic field \mathbf{B} , and check that they satisfy Maxwell's equations in vacuum. You need to state the condition(s) to be imposed on ω and k for your analysis.

- (b) Consider a conducting sphere of radius a and let a charge q be placed at a distance $d > a$ from the center of the sphere (see the figure below). If the potential on the sphere is zero everywhere,



prove that this is equivalent to placing a charge $q' = -\frac{a}{d}q$ at a distance $b = \frac{a^2}{d}$ from the centre of the sphere along the line joining the center and the position of q .

- (c) A substance shows a Raman line at 4567 \AA when the exciting line 4358 \AA is used. Deduce the positions of Stokes and anti-Stokes lines for the same substance when the exciting line 4047 \AA is used.

- P16. (a) Consider the effective intensity of the gravitational force of the sun acting on a particle of mass m to be

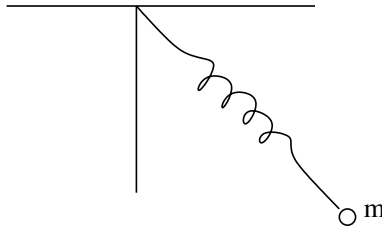
$$F_r = -\frac{k}{r^2} - br,$$

where k and b are constants, and r is the radial distance of the particle from the sun. If the particle moves in a circular orbit of radius r_0 with an angular momentum l , derive a relation between l and r_0 .

- (b) An olympic diver of mass m begins his descent from a 10 metre high diving board with zero initial velocity.

- i. Calculate the velocity V_0 on impact with the water.
- ii. Assume that the buoyant force balances the gravitational force on the diver, and the viscous force of the water on the diver is bv^2 , where v is the velocity of the diver and b is a constant. Set up the equation of motion for vertical descent of the diver through water. Solve for the velocity of the diver as a function of depth x under water with the boundary condition $v = V_0$ at $x = 0$.

- P17. (a) A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k . The other end of the spring is tied to a fixed support (see figure). When no weight is



attached to the spring, its length is l . Assume that the motion of the spring is confined to a vertical plane.

- i. Derive the equation of motion of the mass m .
 - ii. Solve the equations of motion assuming that angular and radial displacements are small.
- (b) In a laboratory experiment, a muon is observed to travel 800 m before it disintegrates. Based upon the lifetime of a muon being 2×10^{-6} s, a student concludes that its speed was

$$v = \frac{800 \text{ m}}{2 \times 10^{-6} \text{ s}} = 4 \times 10^8 \text{ m/s.}$$

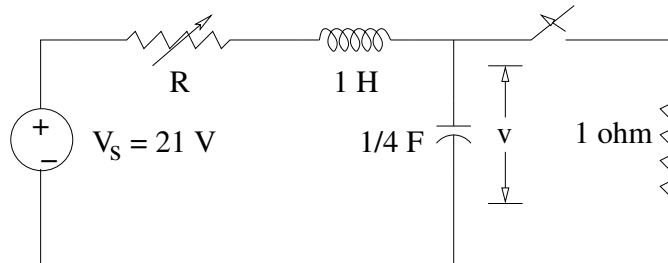
Faster than light! Identify the student's error and find the actual speed of the muon.

- P18. (a) The molar internal energy of a monatomic gas obeying van der Waal's equation is given by $\frac{3}{2}RT - \frac{a}{V}$, where a is a constant, and R , T and V carry their usual meanings.

- i. If the gas is allowed to expand adiabatically into vacuum (i.e., free expansion) from volume V_1 to V_2 , and T_1 is its initial temperature, what is the final temperature of the gas?
 - ii. What would be the final temperature for an ideal gas?
- (b) A single classical particle of mass m with energy $\epsilon \leq E$, where $E = P^2/2m$, is enclosed in a volume V ; P being the momentum corresponding to the energy E .
- i. Determine the (asymptotic) number of accessible microstates in the energy range ϵ to $\epsilon + d\epsilon$.
 - ii. Using this result, obtain the partition function of the aforesaid system if the energy varies between 0 and ∞ .
 - iii. What would be the free energy of the particle?

- P19. (a) An LCR circuit with a variable resistance R is connected across a fixed supply voltage V_s . Initially the switch was closed, and it was opened at time $t = 0$. Find the potential difference v (as indicated in the diagram) and current i in the circuit as a function of time t , when
- i. $R = 4\Omega$ and
 - ii. $R = 5\Omega$.

Also describe the nature of response for both these cases.



- (b) Calculate the energy, in terms of kT and E_F , for which the difference between the Maxwell-Boltzmann (MB) approximation and

the Fermi-Dirac distribution (FD) function is five percent of the FD function. The symbols mentioned carry their usual meanings.

- (c) A beam of electrons with kinetic energy 1 keV is diffracted as it passes through a polycrystalline metal foil. The metal has a cubic lattice structure with a spacing of 1 Å. Calculate the Bragg angle for the first order diffraction maximum.

- P20. (a) The free electron model for a metal assumes that the conduction electrons can be approximated by a gas of free electrons, where the only important parameters for the gas are n and τ . The terms n and τ stand for the number density of electrons and the time between collisions respectively. Show that in this model, the electrical conductivity of the metal can be expressed as:

$$\sigma = \frac{ne^2\tau}{m}$$

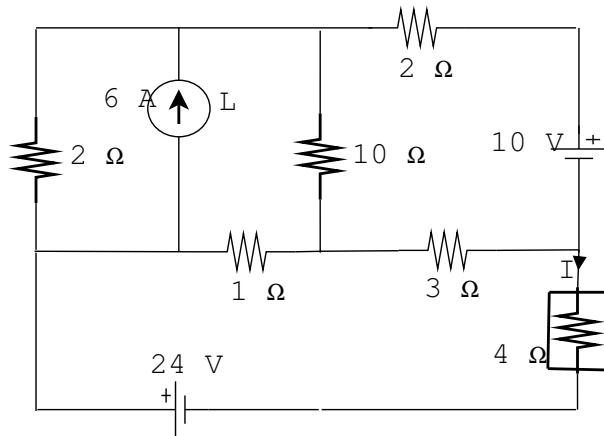
Estimate τ for an electron in copper. The resistivity of copper metal is 1.7×10^{-6} ohm-cm and the atomic density of copper is 8.5×10^{22} atoms/cm³. Assume $e = 4.8 \times 10^{-10}$ e.s.u. and $m = 9.1 \times 10^{-28}$ gm.

- (b) The ground state of chlorine is ${}^2P_{\frac{3}{2}}$. Find its magnetic moment. How many substates will the ground state split into in a weak magnetic field?

(iv) RADIOPHYSICS/TELECOMMUNICATIONS/ELECTRONICS/
ELECTRICAL ENGINEERING

R1. Design a sequential machine that produces an output 1 whenever a substring of 5 consecutive symbols in the input starts with two 1's and contains exactly three 1's. If a substring of 5 symbols starts with two 1's, the analysis of the next substring does not begin until the processing of the current substring is complete. Realize this circuit with the minimum number of NAND gates and flip flops.

R2. In the following circuit, determine the current through the 4Ω resistor. It is assumed that the two batteries are ideal and L is an ideal current source.

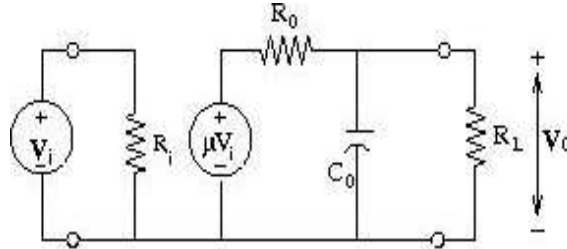


R3. A 50 HP (37.3 KW), 460 V DC shunt motor running freely takes a current of 4 A and runs at a speed of 660 rpm. The resistance of the armature circuit (including brushes) is 0.3 ohm and that of the shunt field circuit is 270 ohm.

- Determine (i) the total current, and (ii) the speed of the motor when it is running at full load.
- Determine the armature current at which the efficiency is maximum (ignore the effect of armature reaction).

R4. Consider a voltage amplifier circuit shown in the figure below, where R_i and R_o represent the input and output impedances respectively,

C_0 is the total parasitic capacitance across the output port, μ is the amplifier gain and the output is terminated by a load resistance R_L .



- (a) Calculate the current, voltage and power gain in decibels (dB) of the amplifier, when
 $R_i = 1M\Omega$, $R_L = 600\Omega$, $R_0 = 100M\Omega$, $C_0 = 10pf$, $\mu = 10$.
- (b) Calculate the 3-dB cutoff frequency of the amplifier when
 $R_i = 5K\Omega$, $R_L = 1K\Omega$, $R_0 = 100\Omega$, $C_0 = 10pf$, $\mu = 2$.

R5. Open-circuit and short-circuit tests are conducted on a 220/440 V, 4.4 KVA, single phase transformer. The following readings are obtained.

Open Circuit test with voltage applied on low-voltage side:

Voltage = 110 V,

Current = 1.1A, and

Power = 150 W.

Short Circuit test with voltage applied on high-voltage side:

Voltage = 24 V,

Current = 8A, and

Power = 64 W.

At 0.8 p.f. lagging, calculate

- (i) the efficiency of the transformer at full-load, and
- (ii) the output voltage at the secondary terminal when supplying full-load secondary current.

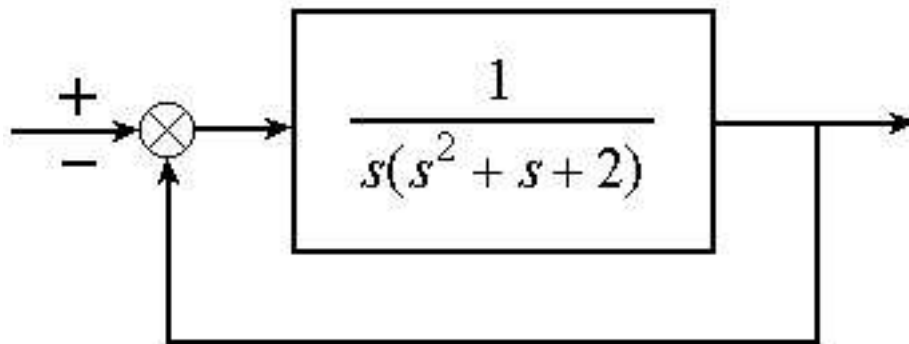
R6. Assume that an analog voice signal which occupies a band from 300 Hz to 3400 Hz, is to be transmitted over a Pulse Code Modulation (PCM) system. The signal is sampled at a rate of 8000 samples/sec. Each sample value is represented by 7 information bits plus 1 parity

bit. Finally, the digital signal is passed through a raised cosine roll-off filter with the roll-off factor of 0.25. Determine

- whether the analog signal can be exactly recovered from the digital signal;
- the bit duration and the bit rate of the PCM signal before filtering;
- the bandwidth of the digital signal before and after filtering;
- the signal to noise ratio at the receiver end (assume that the probability of bit error in the recovered PCM signal is zero).

R7. An abrupt p - n junction is made of silicon where the resistivities of the two sides are ρ_n (n -side) and ρ_p (p -side) respectively. Derive expressions for the contact potential across the junction and the total width of the depletion region for zero applied voltage.

R8. Determine the stability of the following closed loop system. S represents Laplace operator.



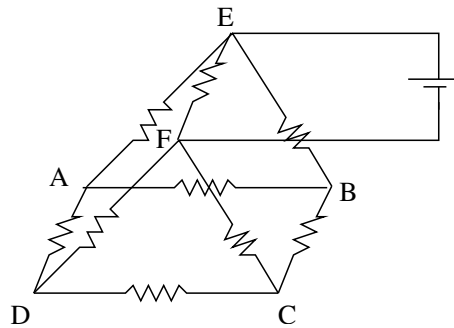
R9. A p -type germanium bar having a rectangular cross-section with sides 3 mm and 5 mm, has an acceptor impurity density of $1.5 \times 10^{15} \text{ cm}^{-3}$. It is used in a Hall effect experiment so that the current flows along the length of the bar with a current density of 1600 amp/m^2 . A magnetic induction field of magnitude 0.5 weber/m^2 is applied along the 5 mm side of the cross-section of the bar. Calculate the Hall voltage developed. Also, derive the necessary formula for this computation.

R10. Consider a hypothetical three-input H gate defined as

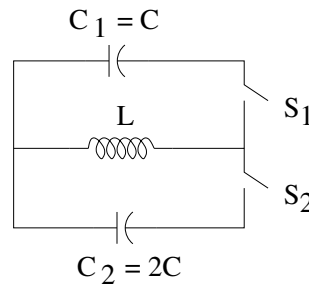
$$H(A, B, C) = AC' + B'C.$$

Using minimum number of H gates, realize the conventional Boolean NOT, two-input AND, two-input OR and two-input XOR. The two input variables may be denoted by A and B .

- R11. (a) A prism is made of wire mesh with each side having equal resistance R . (See the figure given below.) A battery of 6 V and zero internal resistance is connected across E and F. If R is 0.5Ω , find the current in the branch AD.



- (b) In the circuit given below, the capacitor C_1 having capacitance C is charged to potential V_0 . The switch S_1 is closed at time $t = 0$. After time $t = \frac{\pi\sqrt{LC}}{6}$, switch S_1 is opened while switch S_2 is closed. If initially both switches were open and the capacitor C_2 of capacitance $2C$ was uncharged, find the maximum charge stored on C_2 .



- R12. Suppose a sequence of information is composed of five possible symbols with frequencies given in the table below:

Symbol	x_0	x_1	x_2	x_3	x_4
Frequency	100	50	50	200	100

- (a) Encode the above set of symbols with strings of 0's and 1's based on Huffman coding.
- (b) Calculate the average code-word length and entropy of the source.

R13. A 5 KVA distribution transformer has a full load efficiency at unity power factor of 95%. It is given that Copper and Iron losses of the transformer are of equal proportion (1:1) at full load. Calculate the average efficiency per day and explain the steps involved in the computation, if it is loaded throughout 24 hours, and with the following load pattern

- (a) 1/4 th of full load for 7 hours
- (b) No load for 10 hours
- (c) 1/2 of full load for 5 hours
- (d) Full load for 2 hours

Assume the power factor of the load for the whole day to be unity.

R14. (a) A sequence $x[n]$ of length 8 has a Discrete Fourier Transform (DFT) $X[k]$ as follows:

$$X[k] = \begin{cases} 1; & k = 0, 1, 2, 3 \\ k - 2; & k = 4, 5, 6, 7 \\ 0; & \text{otherwise.} \end{cases}$$

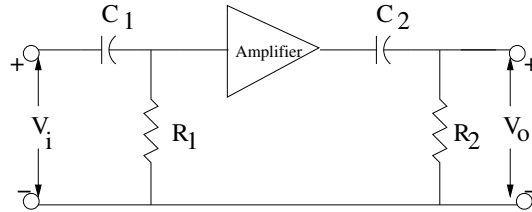
$x[n]$ is now up-sampled by a factor of 2 by inserting a zero between every pair of samples of $x[n]$, and appending a zero at the end to produce a sequence $y[n]$ of length 16. Sketch $Y[k]$, the 16-point DFT of $y[n]$.

(b) A zero-mean, wide-sense stationary sequence with variance σ_x^2 passes through a stable and causal all-pass filter with z-transform

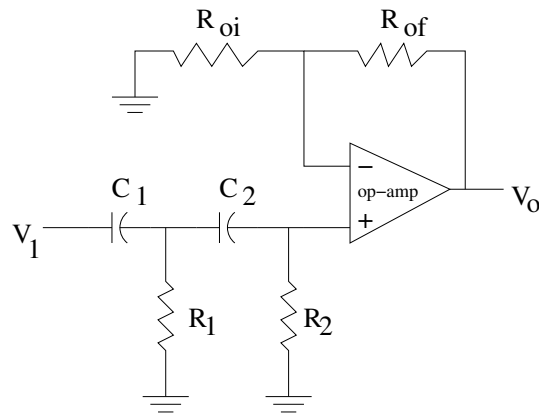
$$H(z) = \frac{\alpha - z^{-1}}{1 - \alpha z^{-1}}.$$

Determine the range of values that α can take. Also calculate the power spectrum of the output, i.e., the Discrete Time Fourier Transform (DTFT) of the auto-correlation sequence of the output.

- R15. (a) Consider the following circuit where the amplifier has the voltage gain of A with zero phase shift. Let a ramp voltage $V_i = \alpha t$, $\alpha (> 0)$ being a constant, be applied to the input. Derive an expression for the output voltage V_o . Also, find the initial slope of V_o as time $t \rightarrow 0$.



- (b) Calculate the gain and cutoff frequency of the following circuit when $R_1 = R_2 = 2 \text{ k}\Omega$, $C_1 = C_2 = 0.05 \text{ }\mu\text{F}$, $R_{oi} = 10 \text{ k}\Omega$ and $R_{of} = 50 \text{ k}\Omega$.



(v) COMPUTER SCIENCE

- C1. (a) Write the smallest real number greater than 6.25 that can be represented in the IEEE-754 single precision format (32-bit word with 1 sign bit and 8-bit exponent).
- (b) Convert the sign-magnitude number **10011011** into a 16-bit 2's complement binary number.
- (c) The CPU of a machine is requesting the following sequence of memory accesses given as word addresses: 1, 4, 8, 5, 20, 17, 19, 56. Assuming a direct-mapped cache with 8 one-word blocks, that is initially empty, trace the content of the cache for the above sequence.
- C2. Consider a collection of n binary strings S_1, \dots, S_n . Each S_i is of length l_i bits where $1 \leq l_i \leq k$.
- (a) Write a function **prefix(S,T)** in C programming language that takes two binary strings S and T and returns 1 if S is a prefix of T , else it returns 0. For example, **prefix(001,00101)** returns 1 but **prefix(010,00101)** returns 0.
- (b) Suppose we want to report all the pairs (i, j) for which S_i is a prefix of S_j , ($1 \leq i \neq j \leq n$). How many times do we need to call the **prefix** function described above?
- (c) Present an $O(nk)$ time algorithm to report all the (i, j) 's as mentioned in (b). (Hint: Use a binary tree with each edge marked as 0 or 1; a path from the root to a node in the tree represents a binary string.)
- C3. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of n integers. A pair (x_i, x_j) is said to be the closest pair if $|x_i - x_j| \leq |x'_i - x'_j|$, for all possible pairs (x'_i, x'_j) , $i', j' = 1, 2, \dots, n, i' \neq j'$.
- (a) Describe a method for finding the closest pair among the set of integers in S using $O(n \log_2 n)$ comparisons.
- (b) Now suggest an appropriate data structure for storing the elements in S such that if a new element is inserted to the set S or an already existing element is deleted from the set S , the current closest pair can be reported in $O(\log_2 n)$ time.
- (c) Briefly explain the method of computing the current closest pair, and necessary modification of the data structure after each update. Justify the time complexity.

C4. Consider the C program given below.

```
#include <stdio.h>
#define n 5
#define m n*n
int **a;

main(){
    int i, j, k;
    void printarray();
    a = (int **)calloc(n, sizeof(int *));
    for(i = 0; i < n; i++)
        *(a+i) = (int *)calloc(n, sizeof(int));

    for(k=1, i=0, j=(n-1)/2; k<=m; i--, j--, k++){
        i = (i + m) % n;
        j = (j + m) % n;
        if (a[i][j] != 0){
            i = (i + 2) % n;
            j = (j + 1) % n;
        }
        a[i][j] = k;
    }
    printarray();
    return 0;
}

void printarray(){
    int i, j;

    for(i = 0; i < n; i++){
        for (j = 0; j < n; j++)
            printf("%d", a[i][j]);
        printf("\n");
    }
}
```

- (a) Write the output of the above program.
- (b) Modify the subroutine “printarray” to print all the elements of the array in decreasing order.

C5. Let $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 2$.

- (a) Express 63 as a sum of distinct a_i 's.
- (b) Write an algorithm to express any positive integer k as a sum of at most $\lceil \log_2 k \rceil$ many distinct a_i 's.
- (c) Prove the correctness of your algorithm.

C6. Let A be an $n \times n$ matrix such that for every 2×2 sub-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A , if $a < b$ then $c \leq d$. Note that for any pair of rows i and j , if a_{ik} and a_{jl} are the largest elements in i -th and j -th rows of A , respectively, then $k \leq l$ (as illustrated in the 5×5 matrix below).

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 1 \\ 7 & 8 & 5 & 6 & 4 \\ 2 & 3 & 6 & 6 & 5 \\ 5 & 6 & 9 & 10 & 7 \\ 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

- (a) Write an algorithm for finding the maximum element in each row of the matrix with time complexity $O(n \log n)$.
- (b) Establish its correctness, and justify the time complexity of the proposed algorithm.

C7. Consider a file consisting of 100 blocks. Assume that each disk I/O operation accesses a complete block of the disk at a time. How many disk I/O operations are involved with contiguous and linked allocation strategies, if one block is

- (a) added at the beginning?
- (b) added at the middle?
- (c) removed from the beginning?
- (d) removed from the middle?

C8. (a) Consider the context-free grammar $G = (\{S, A\}, \{a, b\}, S, P)$, where

$$P = \{S \rightarrow AS, \\ S \rightarrow b, \\ A \rightarrow SA, \\ A \rightarrow a\}$$

Show that G is left-recursive. Write an equivalent grammar G' free of left-recursion.

- (b) Consider the grammar $G = (\{S, T\}, \{a, \pi, (,), +\}, S, P)$, where

$$P = \begin{cases} S \rightarrow a|\pi|(T), \\ T \rightarrow T + S|S \end{cases}$$

Find the parse tree for the sentence:

$$(((a + a) + \pi + (a)) + a)$$

- C9. (a) Five batch jobs P_1, \dots, P_5 arrive almost at the same time. They have estimated run times of 10, 6, 2, 4 and 8 ms. Their priorities are 3, 5, 2, 1 and 4 respectively, where 1 indicates the highest priority and 5 indicates the lowest. Determine the average turnaround and waiting time for the following scheduling algorithms:
- (i) Round robin with time quantum of 5 ms,
 - (ii) Priority scheduling.
- (b) The access time of a cache memory is 100 ns and that of main memory is 1000 ns. It is estimated that 80% of the memory requests are for read and the remaining 20% are for write. The hit ratio for read access is 0.9. A write through procedure is used.
- (i) What is the average access time of the system considering only memory read cycles?
 - (ii) What is the average access time of the system considering both read and write requests?
- C10. (a) A program P consisting of 1000 instructions is run on a machine at 1 GHz clock frequency. The fraction of floating point (FP) instructions is 25%. The average number of clock-cycles per instruction (CPI) for FP operations is 4.0, and that for all other instructions is 1.0.
- (i) Calculate the average CPI for the overall program P .
 - (ii) Compute the execution time needed by P in seconds.
- (b) Consider a 100Mbps token ring network with 10 stations having a ring latency of 50 μ s (the time taken by a token to make one complete rotation around the network when none of the stations is active). A station is allowed to transmit data when it receives the token, and it releases the token immediately after transmission. The maximum allowed holding time for a token (THT) is 200 μ s.

- (i) Express the maximum efficiency of this network when only a single station is active in the network.
 - (ii) Find an upper bound on the token rotation time when all stations are active.
 - (iii) Calculate the maximum throughput rate that one host can achieve in the network.
- C11. Consider a graph G (called an interval graph) whose nodes correspond to a set of intervals on the real line. The i -th interval is denoted by $[\alpha_i, \beta_i]$, where $0 \leq \alpha_i < \beta_i$. An edge between two nodes (i, j) implies that the corresponding intervals $[\alpha_i, \beta_i]$ and $[\alpha_j, \beta_j]$ overlap.
- (a) Consider the set of intervals $[3, 7]$, $[2, 4]$, $[2, 3]$, $[1, 5]$, $[1, 2]$, $[6, 7]$, $[10, 16]$, $[11, 12]$. Draw the corresponding interval graph and identify the largest subgraph where all the nodes are connected to each other.
 - (b) Write an algorithm which takes the interval graph G as input and finds the largest subgraph of G in which all the nodes are connected to each other. What is the time complexity of your algorithm?
 - (c) Given a list of intervals, write an algorithm to list all the connected components in the corresponding interval graph. What is the time complexity of your algorithm?
- C12. (a) Let $R = (A, B, C, D, E, F)$ be a schema with the set $F = \{A \rightarrow BC, CD \rightarrow E, B \rightarrow D, E \rightarrow A\}$ of functional dependencies. Suppose R is decomposed into two schema $R_1 = (A, B, C)$ and $R_2 = (A, D, E, F)$
- (i) Is this decomposition loss-less? Justify.
 - (ii) Is this decomposition dependency preserving? Justify.
 - (iii) Identify all the candidate keys for R .
 - (iv) Decompose R into normalized sets of relations using 3NF.
 - (v) If a new dependency $A \twoheadrightarrow F$ (multi-valued dependency) is introduced, what would be the new set of normalized relations?
- (b) Consider the relations $r1(A, B, C)$, $r2(C, D, E)$ and $r3(E, F)$. Assume that the set of all attributes constitutes the primary keys of these relations, rather than the individual ones. Let $V(C, r1)$ be 500, $V(C, r2)$ be 1000, $V(E, r2)$ be 50, and $V(E, r3)$ be 150, where $V(X, r)$ denotes the number of distinct values that appear

in relation r for attribute X . If r_1 has 1000 tuples, r_2 has 1500 tuples, and r_3 has 750 tuples, then give the ordering of the natural join $r_1 \bowtie r_2 \bowtie r_3$ for its efficient computation. Justify your answer.

- C13. (a) (i) Write a Context Free Grammar (CFG) for structure definitions in C . Assume that the only allowable types are char, int, and float (you need not handle pointers, arrays, structure, fields, etc.).
- (ii) Assume that chars are stored using 1 byte each; ints and floats are stored using 4 bytes each and are aligned at 4 byte boundaries. Add semantic rules to your grammar to calculate the number of bytes required to store the structure defined by your grammar.
- (b) (i) Compute the canonical collection of sets of $LR(1)$ items (*i.e.* canonical LR items) for the following grammar: $S \rightarrow aXcd$, $S \rightarrow aYce$, $X \rightarrow b$, $Y \rightarrow b$. Is the grammar $LR(1)$? Briefly justify.
- (ii) Give an example of a grammar that is unambiguous but not $LR(2)$. Briefly justify/explain your example.

C14. An operating system allocates memory in units of 1 KB pages. The address space of a process can be up to 64 MB in size; however, at any point of time, a process can be allocated at most 16 MB of physical memory. In addition the kernel uses 65 KB of physical memory to store page table entries of the current process. The OS also uses a translation-lookaside buffer (TLB) to cache *page table entries*. You are also given the following information:

- size of a page table entry is 4 bytes,
- TLB hit ratio is 90%,
- time for a TLB lookup is negligible,
- time for a memory read is 100 nanoseconds,
- time to read a page from the swapping device into physical memory is 10 milliseconds.

Calculate the effective memory access time for a process whose address space is 20 MB? Assume that memory accesses are random and distributed uniformly over the entire address space.

- C15. (a) What are the conditions which must be satisfied by a solution to the critical section problem?
- (b) Consider the following solution to the critical section problem for two processes. The two processes, P_0 and P_1 , share the following variables:

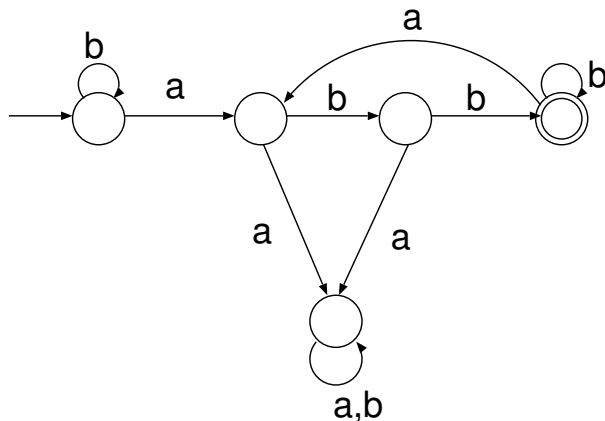
```
var flag : array [0..1] of Boolean;
                                (* initially false *)
    turn : 0..1;
```

The program below is for process P_i ($i = 0$ or 1) with process P_j ($j = 1$ or 0) being the other one.

```
repeat
  flag[i] <- true;
  while (flag[j])
    do if (turn = j)
      then begin
          flag[i] <- false;
          while (turn = j) do skip;
        end;
    ...
    CRITICAL SECTION
    ...
  turn <- j;
  flag[i] <- false;
  ...
  REMAINDER SECTION
  ...
until false;
```

Does this solution satisfy the required conditions?

- C16. (a) Construct an AVL tree of height 5 with minimum number of nodes.
- (b) Consider a B-tree of order 3.
- (i) Trace the insertion of the keys a, g, f, b, k, d, h, m , into an initially empty tree, in lexicographic order.
 - (ii) Sketch the B-tree upon deletion of keys h, d .
- C17. (a) Construct a finite state machine that accepts all the binary strings in which the number of 1's and number of 0's are divisible by 3 and 2, respectively.



- (b) Describe the language recognized by the following machine.
- (c) Consider the grammar $E \rightarrow E + n|E \times n|n$. For a sentence $n + n \times n$, find the handles in the right sentential form of the reductions.
- C18. (a) Station A is sending data to station B over a full duplex error free channel. A sliding window protocol is being used for flow control. The send and receive window sizes are 6 frames each. Each frame is 1200 bytes long and the transmission time for such a frame is $70 \mu\text{S}$. Acknowledgment frames sent by B to A are very small and require negligible transmission time. The propagation delay over the link is $300 \mu\text{S}$. What is the maximum achievable throughput in this communication?
- (b) Consider a large number of hosts connected through a shared communication channel. The hosts transmit whenever they have any data. However, if two data packets try to occupy the channel at the same time, there will be a collision and both will be garbled. The hosts retransmit these packets that suffered collisions. Assume that the generation of new packets by a host is a Poisson process and is independent of other hosts. The total number of transmissions (old and new packets combined) per packet time follows Poisson distribution with mean 2.0 packets per packet time. Compute the throughput of the channel. (Packet time is the amount of time needed to transmit a packet.)
- C19. For a graph $G = (V, E)$, a vertex coloring with x colors is defined as a function $C : V \rightarrow \{1, 2, \dots, x\}$, such that $C(u) \neq C(v)$ for every edge

$(u, v) \in E$. The chromatic polynomial $P(G, x)$ denotes the number of distinct vertex colorings for G with at most x colors. For example, $P(K_2, x) = x(x - 1)$.

(a) Derive the chromatic polynomial for the graphs

i. $K_{1,5}$

ii. C_4

where $K_{m,n}$ denotes the complete bipartite graph of m and n vertices in the two partitions, and C_n is the simple cycle with n vertices.

(b) Consider K_n to be a complete graph with n vertices. Express $P(K_n, x)$ as a recurrence relation and solve for it.

C20. The diameter of a tree $T = (V, E)$ is given by $\max_{u,v \in V} \{\delta(u, v)\}$, where $\delta(u, v)$ is the shortest path distance (i.e., the length of the shortest path) between vertices u and v . So, the diameter is the largest of all shortest path distances in the tree.

(a) Give the pseudo-code for an efficient algorithm to compute the diameter of a given tree T .

(b) Analyze the time complexity of your algorithm.

(c) What is its space complexity?

(d) Clearly mention the data structure(s) used by your algorithm.

(e) A vertex c is called a center of a tree T if the distance from c to its most distant vertex is minimum among all vertices in V . Write an algorithm to determine and report a center of the given tree T .

C21. (a) Write a context-free grammar for the language consisting of all strings over $\{a, b\}$ in which the number of a 's is not the same as that of b 's.

(b) Let the set $D = \{p \mid p \text{ is a polynomial over the single variable } x, \text{ and } p \text{ has a root which is an integer (positive or negative)}\}$.

i. Design a Turing machine (TM) to accept D .

ii. Can D be decided by a TM? Justify your answer.