JRF IN MATHEMATICS 2006-2007

There will be two tests RM-1, and RM-2 of 2 hours duration each in the forenoon and in the afternoon. Topics to be covered in these tests along with an outline of the syllabus and sample questions are given below:

1) Topics for MI (Forenoon examination) : Real Analysis, Measure and Integration, Complex Analysis, Ordinary Differential Equations and General Topology.

2) Topics for MII (Afternoon examination) : Algebra, Linear Algebra, Functional Analysis, Elementary Number Theory and Combinatorics.

Candidates will be judged based on their performance in **both** the tests.

OUTLINE OF THE SYLLABUS

1. **General Topology :** Topological spaces, Continuous functions, Connectedness, Compactness, Separation Axioms. Product spaces. Complete metric spaces. Uniform continuity.

2. **Functional Analysis :** Normed linear spaces, Banach spaces, Hilbert spaces, Compact operators. Knowledge of some standard examples like $C[0,1], L^p[0,1]$. Continuous linear maps (linear operators). Hahn-Banach Theorem, Open mapping theorem, Closed graph theorem and the uniform boundedness principle.

3. **Real analysis :** Sequences and series, Continuity and differentiability of real valued functions of one and two real variables and applications, uniform convergence, Riemann integration.

4. **Linear algebra :** Vector spaces, linear transformations, characteristic roots and characteristic vectors, systems of linear equations, inner product spaces, diagonalization of symmetric and Hermitian matrices, quadratic forms.

5. **Elementary number theory :** Divisibility, congruences, standard arithmetic functions, permutations and combinations.

6. **Lebesgue integration :** Lebesgue measure on the line, measurable functions, Lebesgue integral, convergence almost everywhere, monotone and dominated convergence theorems.

7. **Complex analysis :** Analytic functions, Cauchy's theorem and Cauchy integral formula, maximum modulus principle, Laurent series, Singularities, Theory of residues, contour integration.

8. **Abstract algebra :** Groups, Symmetric and Alternating groups, Direct product and finite abelian groups, Sylow theorems; rings, polynomial rings, integral domains, Euclidean rings; fields, extension fields, roots of polyno-

mials, finite fields.

9. **Ordinary differential equations :** First order ODE and their solutions, singular solutions, initial value problems for first order ODE, general theory of homogeneous and nonhomogeneous linear differential equations.

SAMPLE QUESTIONS

1. Let $\{x_n\}_{n=1}$ be a bounded sequence of real numbers. Show that there exists a real number α and positive integers $n_1, n_2, \cdots$ such that $n_1 < n_2 < \cdots$ and $\sum_k |x_{n_k} - \alpha| < \infty$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function such that f and its derivative have no common zero in the closed interval $[0, 1]$. Show that f cannot have infinitely many zeroes in $[0, 1]$.

3. Let a, b, c, d be real numbers. Show that the zeroes of the functions $f(x) = a \cos x + b \sin x$ and $g(x) = c \cos x + d \sin x$ are distinct and alternate whenever $ad - bc \neq 0$.

4. Let f be a real valued continuous periodic function on the real line R with period p. Show that $F(x) = \int_0^x f(t)\, dt$ is periodic with period p if and only if $\int_0^p f(t)\, dt = 0$.

5. Let S denote the set of all real numbers $0 \leq x \leq 1$ such that the digit 6 never occurs in the decimal expansion of x. Show that S is a Borel set and has Lebesgue measure zero.

6. Show that the (continuous) real valued function on the complex plane taking a complex number to its real part is open but not closed. (Recall that a map is called open(respectively, closed) if the image of every open(respectively, closed) set is open(respectively, closed).)

7. For any $\epsilon > 0$, show that the sum of the rational numbers in the interval $(0, \epsilon)$ is infinite.

8. For $f \in L^1[0, 2\pi]$ define $\hat{f}(n)$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$$

for $n = 0, \pm 1, \pm 2, \ldots$. Suppose that X is a closed linear subspace of $L^1[0, 2\pi]$ such that $\sum_n |\hat{f}(n)| < \infty$ for each $f \in X$. Show that there is a constant $M < \infty$ such that $\sum_n |\hat{f}(n)| \leq M \int_0^{2\pi} |f(t)| dt$ for each $f \in X$. [Hint: Use closed graph theorem].

9. Let $q(X)$ be a polynomial in X of degree n with real coefficients and let k be a non-zero real number. Show that the differential equation

$$\frac{dy}{dx} + ky(x) = q(x)$$

has exactly one polynomial solution of degree n.

10. Let a, b, c be real numbers with $a > 0$, $c > 0$. Show that every solution of the equation

$$\frac{dy}{dt} + ay = be^{-ct}$$

approaches zero as t approaches ∞.

11. Let H be a Hilbert space and $S \subseteq H$ be a finite set. Show that $(S^{\perp})^{\perp}$ is a finite dimensional vector space.

12. Let x_1, x_2, x_3, x_4 be vectors in $\mathbb{R}^n$ such that the inner products $\langle x_i, x_j \rangle$, $i \neq j$, are strictly negative. Show that any three of the vectors x_1, x_2, x_3, x_4 are linearly independent.

13. Let $y_1, y_2, \ldots$ be a sequence in a Hilbert space. Let V_n be the linear span of $\{y_1, y_2, \ldots, y_n\}$. Assume that $\|y_{n+1}\| \leq \|y - y_{n+1}\|$ for each $y \in V_n$ for $n = 1, 2, 3, \ldots$. Show that $\langle y_i, y_j \rangle = 0$ for $i \neq j$.

14. Show that for $0 < a < 1$,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} = \frac{1}{1 - a^2}.$$

(Hint: Take $z = e^{i\theta}$ and apply Cauchy's formula.)

15. Let $f(z), z = x + iy$, be an analytic function and u be its real part. If u is a polynomial in the variables x and y, then show that $f(z)$ is a polynomial in z.

16. Show that there is no polynomial P with complex coefficients such that $P(n) = (-1)^n$ for all integers n. Does there exist an entire function with this property ?

17. Find the number of elements of order 6 in the permutation group S_6 on six symbols.

18. If m and n are distinct natural numbers, then show that the abelian groups $\mathbb{Z}^m$ and $\mathbb{Z}^n$ are not isomorphic.

19. Let p and q be prime numbers. If q divides $2^p - 1$ then show that p divides $q - 1$.

20. Let σ be a permutation of $\{1, 2, 3, \dots, n\}, n$ odd. Show that

$$(\sigma(1) - 1)(\sigma(2) - 2) \dots (\sigma(n) - n)$$

is even.

21. Give two non-isomorphic field extensions of degree 2 of the field Q of rational numbers. Justify your answer.

22. Can the ideal $I = \left\{ f : [0,1] \to \mathbb{R},\ f\left(\frac{1}{2}\right) = 0 \right\}$ in the ring R of all functions from $[0,1]$ to $\mathbb{R}$ be generated by one element?

23. Find all ring automorphisms of $Q[X]$ where Q denotes the field of rational numbers.

24. Does there exist an integer x satisfying the following congruences ?

$$10x \equiv 1 \ mod\ 21$$

$$5x \equiv 2 \ mod\ 6$$

$$4x \equiv 1 \ mod\ 7$$

25. Let

$$M = \{(x, y) \in \mathbf{R}^2 : x, y \in \mathbf{Z}\}.$$

Let L be the line in $\mathbf{R}^2$ defined by the equation $ax + by = c$, where a, b, c are non-zero integers such that a and b are co-prime. Show that $L \cap \mathsf{M}$ is non-empty and find the least distance between two distinct points of $L \cap \mathsf{M}$.

26. Find an $n \times n$ matrix over real numbers whose minimal polynomial is x^{n-1}.

27. Find a maximal linearly independent set in the following subset $\mathcal{C}$ of $\mathbf{R}^{k+1}$:

$$\mathcal{C} = \{(1, t, t^2, \ldots, t^k) \in \mathbf{R}^{k+1} : t \in \mathbf{R}\}.$$

28. Suppose A, B, C are three $n \times n$ matrices such that A has n distinct eigenvalues. If $AB = BA$ and $AC = CA$, prove that $BC = CB$.

29. Let C be an invertible, real matrix of order n. If each row sum of C is 1, then show that each row sum of C^{-1} also is 1.

30. Let n be a positive integer and p a prime. Show that the number of ordered basis of a vector space of dimension n over a field of order p is

$$(p^n - 1)(p^n - p)(p^n - p^2) \ldots (p^n - p^{n-1}).$$

Deduce that this number is a multiple of $n!$.