

Test Code : RC (Short Answer Type) 2008

JRF in Computer and Communication Sciences

The Candidates for Junior Research Fellowship in Computer Science and Communication Sciences will have to take two tests - Test MIII (objective type) in the forenoon session and Test RC (short answer type) in the afternoon session. The RC test booklet will have two groups as follows:

GROUP A

A test for all candidates in logical reasoning and basics of programming, carrying 20 marks.

GROUP B

A test, divided into five sections, carrying equal marks of 80 in the following areas at M.Sc./M.E./M.Tech. level:

(i) Mathematics, (ii) Statistics, (iii) Physics, (iv) Radiophysics/ Telecommunication/ Electronics/ Electrical Engg., and (v) Computer Science.

A candidate has to answer questions from *only* one of these sections, according to his/her choice.

The syllabi and sample questions of the RC test are given overleaf.

Syllabus

Elements of Computing:

Logical reasoning, basics of programming (using pseudo-codes), Elementary data types and arrays.

Mathematics:

Graph theory and combinatorics: Graphs and digraphs, paths and cycles, trees, Eulerian graphs, Hamiltonian graphs, chromatic numbers, planar graphs, tournaments, inclusion-exclusion principle, pigeon-hole principle.

Linear programming: Linear programming, simplex method, duality.

Linear algebra: Vector spaces, basis and dimension, linear transformations, matrices, rank, inverse, determinant, systems of linear equations, characteristic roots (eigen values) and characteristic vectors (eigen vectors), orthogonality and quadratic forms.

Abstract algebra: Groups, subgroups, cosets, Lagrange's theorem, normal subgroups and quotient groups, permutation groups, rings, subrings, ideals, integral domains, fields, characteristic of a field, polynomial rings, unique factorization domains, field extensions, finite fields.

Elementary number theory: Elementary number theory, divisibility, congruences, primality.

Calculus and real analysis: Real numbers, basic properties, convergence of sequences and series, limits, continuity, uniform continuity of functions, differentiability of functions of one or more variables and applications, indefinite integral, fundamental theorem of calculus, Riemann integration, improper integrals, double and multiple integrals and applications, sequences and series of functions, uniform convergence.

Differential equations: Solutions of ordinary and partial differential equations and applications.

Statistics:

Probability Theory and Distributions: Basic probability theory, discrete and continuous distributions, moments, characteristic functions, Markov chains.

Estimation and Inference: Sufficient statistics, unbiased estimation, maximum likelihood estimation, consistency of estimates, most powerful and uniformly most powerful tests, unbiased tests and uniformly most powerful unbiased tests, confidence sets.

Linear Models: Gauss-Markov set up and least squares theory, multiple linear regression, one and two way analysis of variance.

Multivariate Analysis: Multivariate normal distribution, Wishart distribution, Hotelling's T^2 test, principal component analysis, multiple and canonical correlations, discriminant analysis, cluster analysis, factor analysis.

Physics:

Classical Mechanics: Variational principle and Lagrange's equation, central force problem, rigid body motion, Hamilton equation of motion, canonical transformations, Hamilton Jacobi theory and action angle variables, Lagrangian and Hamiltonian formulation for continuous systems and fields, relativistic mechanics.

Electrodynamics: Electromagnetic fields and potentials, electromagnetic radiation, scattering, dispersion, relativistic electrodynamics.

Thermodynamics and Statistical Mechanics: Reviews of thermodynamics, statistical basis of thermodynamics, density matrix formulation, ensembles, partition function and its uses, Maxwell-Boltzmann, Bose-Einstein and Fermi Dirac statistics, simple gases, Ising model.

Non-Relativistic Quantum Mechanics: Basics of quantum mechanics, the two body problem and central potential, quantum particles in electromagnetic fields, matrix mechanics and spin, approximate methods: stationary states, approximative methods: time dependent problems.

Solid State Physics: Crystal structures, interacting forces, lattice vibrations, electronic band structures, density of states, elementary excitations, transport properties.

Electronics: Basics of semiconductor physics, amplifiers, communication principles.

Vibrations and Waves: Forced vibrations, coupled vibrations, stretched strings, small oscillations.

Radiophysics/Telecommunication/Electronics/Electrical Engg.:

Boolean algebra, digital circuits and systems, circuit theory, amplifiers, oscillators, digital communication, digital signal processing, linear control theory, electrical machines.

Computer Science:

Discrete Mathematics: Elementary counting techniques, Principles of inclusion-

exclusion, recurrence relations, generating functions, propositional logic.

Data Structures: Stack, queue, linked list, binary tree, heap, AVL tree, B-tree.

Design and Analysis of Algorithms: Order notation, sorting, selection, searching, hashing, string handling algorithms, graph algorithms, NP-completeness.

Programming Languages: Fundamental concepts - abstract data types, procedure call and parameter passing, C language.

Computer Organization and Architecture: Number representation, computer arithmetic, memory organization, I/O organization, microprogramming, pipelining, instruction level parallelism.

Operating Systems: Memory management, processor management, critical section, deadlocks, device management, concurrency control.

Formal Languages and Automata Theory: Finite automata and regular expression, context-free grammars, Turing machines, elements of undecidability.

Principles of Compiler Construction: Lexical analyzer, symbol table, parser, code optimization.

Database Systems: Relational model, relational algebra, relational calculus, functional dependency, normalization (upto 3rd normal form).

Computer Networks: Layered network structures, network security, LAN technology - Bus/tree, Ring, Star; data communications - data encoding, flow control, error detection/correction.

Sample Questions

*Note that all questions in the sample set are not of equal difficulty.
They may not carry equal marks in the test.*

GROUP A

ELEMENTS OF COMPUTING

- A1. The strength of a mug (made of glass) is defined as follows.

There is a building having an infinite number of floors. A mug is said to possess strength h units if it does not break when it is dropped from the h -th floor, but it breaks when it is dropped from the ℓ -th floor, where $\ell \geq h + 1$. The strength of a mug is known to be finite.

If you are given only one mug, you can determine its strength by dropping it successively from 1st, 2nd, \dots , floors until it breaks. Thus, if the strength of the mug is h , then the number of times you need to perform the experiment is $h + 1$, where *an experiment consists of dropping the mug from a floor and observing whether it breaks after reaching the ground*. Note that we may use the same mug for many experiments until it breaks.

Now consider that instead of one, you are given two mugs of the *same strength*. Design a scheme to determine the strength of these mugs with minimum number of experiments. Also report the exact number of experiments you have performed in your scheme.

- A2. How many isomers are there for the organic compound C_6H_{14} ? In other words, how many distinct non-isomorphic unlabelled trees are there with 6 vertices of degree 4, and 14 vertices of degree 1?
- A3. In the following table, find the entry in the square marked with *. Justify your answer.

| | | |
|--------|-----------|-----------|
| BD_1 | CE_5 | DF_{21} |
| EG_2 | FH_8 | GI_{34} |
| HJ_3 | IK_{13} | * |

A4. Consider the pseudo-code given below.

Input: Integers b and c .

1. $a_0 \leftarrow \max(b, c)$, $a_1 \leftarrow \min(b, c)$.
2. $i \leftarrow 1$.
3. Divide a_{i-1} by a_i . Let q_i be the quotient and r_i the remainder.
4. If $r_i = 0$ then go to Step 8.
5. $a_{i+1} \leftarrow a_{i-1} - q_i * a_i$.
6. $i \leftarrow i + 1$.
7. Go to Step 3.
8. Print a_i .

What is the output of the above algorithm? What is the mathematical relation between the output a_i and the two inputs b and c ?

A5. Write the output of the following pseudo-code:

1. for ($n = 15$, downto 2, step -2)
2. if ($n > 10$)
3. then $n \leftarrow n + 1$ and print n ;
4. else $n \leftarrow n - 1$ and print n ;
5. endfor

A6. Given an array of n integers, write a pseudo-code for reversing the contents of the array without using another array. For example, for the array

10 15 3 30 3

the output should be

3 30 3 15 10.

You may use one temporary variable.

A7. Consider the sequence $a_n = a_{n-1} a_{n-2} + n$ for $n \geq 2$, with $a_0 = 1$ and $a_1 = 1$. Is a_{2006} odd? Justify your answer.

A8. Derive an expression for the maximum number of regions that can be formed within a circle by drawing n chords.

A9. A function *PrintRec* is defined as follows:

```

PrintRec(A, B, C, n)
  begin
    if n > 0
      begin
        print B;
        PrintRec(B, A, C, n - 1);
        print A;
        PrintRec(C, A, B, n - 2);
      end
    end
  end

```

Find the output for the function call *PrintRec*(x, y, z, 3). Show the intermediate steps of your derivation.

GROUP B

(i) MATHEMATICS

- M1. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function for which there does not exist any $x \in [0, 1]$ such that both $f(x) = 0$ and $f'(x) = 0$. Show that f has only a finite number of zeros in $[0, 1]$.
 $[f'(x)$ denotes the derivative of f at $x]$.
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f'(x)$ exists and is continuous in $[0, 1]$. Show that

$$\lim_{x \downarrow 0} \frac{1}{x^2} \int_0^x (x - 3y)f(y)dy = -\frac{f(0)}{2}.$$

$[x \downarrow 0$ denotes: x decreases to zero.]

- M2. (a) Let $f : [0, 1] \rightarrow [0, 1]$ be such that

$$f(x) = nx - [nx]; \quad \frac{1}{n} < x \leq \frac{1}{n-1}; \quad n = 2, 3, 4, \dots; \quad x \neq 0$$

and $f(0) = 0$. Show that $\int_0^1 f(x)dx$ exists and find its value.

Note: $[y] =$ Largest integer $\leq y$; $y \in \mathbb{R}$.

- (b) Let $f : [0, 1] \rightarrow (0, \infty)$ be continuous. Let

$$a_n = \left(\int_0^1 (f(x))^n dx \right)^{\frac{1}{n}}; \quad n = 1, 2, 3, \dots$$

Find $\lim_{n \rightarrow \infty} a_n$.

- M3. (a) Suppose f is a continuous real valued function on $[0, 1]$. Show that

$$\int_0^1 f(x)xdx = \frac{1}{2}f(\xi)$$

for some $\xi \in [0, 1]$.

- (b) For every $x \geq 0$, prove that

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}},$$

for a unique $\theta(x)$, $0 < \theta(x) < 1$. Prove that, in fact

- (i) $\frac{1}{4} \leq \theta(x) \leq \frac{1}{2}$,
(ii) $\lim_{x \rightarrow 0} \theta(x) = \frac{1}{4}$ and $\lim_{x \rightarrow \infty} \theta(x) = \frac{1}{2}$.

- M4. (a) Show that $f(x) = e^{|x|} - x^5 - x - 2$ has at least two real roots, where e is the base of natural logarithms.
(b) Let $\sum a_n$ be a convergent series such that $a_n \geq 0$ for all n . Show that $\sum \sqrt{a_n}/n^p$ converges for every $p > \frac{1}{2}$.

- M5. (a) Let G be the set of all non-singular 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where the elements a, b, c, d belong to the field of order 3. Using matrix multiplication as the operation in G what is the order of group G ? Is G abelian? Justify your answers.
(b) Prove that $Aut(Q, +) \simeq Z_2$, where Z_2 is the group consisting of only two elements; and $Aut(Q, +)$ is the automorphism group of $(Q, +)$.

- M6. Let $R = (S, +, \cdot, 0, 1)$ be a commutative ring and n be a positive integer such that $n = 2^k$ for some positive integer k .

- (a) Show that for every $a \in S$

$$\sum_{i=0}^{n-1} a^i = \prod_{i=0}^{k-1} (1 + a^{2^i}).$$

- (b) Let $m = w^{\frac{n}{2}} + 1$ where $w \in S$, $w \neq 0$. Show that for $1 \leq p < n$,

$$\sum_{i=0}^{n-1} w^{ip} \equiv 0 \pmod{m}.$$

- M7. (a) Show that there is a basis consisting of only symmetric and skew-symmetric matrices for the vector space of all $n \times n$ real matrices over \mathbb{R} .
- (b) Does there exist a linear transformation from \mathbb{R}^3 to \mathbb{R}^3 such that $(0,1,1)$, $(1,1,0)$, $(1,2,1)$ are mapped to $(1,0,0)$, $(0,1,0)$, $(0,1,1)$ respectively? Justify your answer.
- M8. (a) Let A be a square matrix such that $A^2 = A$. Show that all eigenvalues of A are 0 or 1.
- (b) Let A be a symmetric matrix whose eigenvalues are 0 or 1. Show that $A^2 = A$.
- (c) Suppose A is an $n \times n$ matrix such that $A^2 = A$. Show that $r + s = n$, where $\text{rank}(A) = r$ and $\text{rank}(I - A) = s$.
- M9. Let k be a positive integer. Let $G = (V, E)$ be the graph where V is the set of all strings of 0's and 1's of length k , and $E = \{(x, y) : x, y \in V, x \text{ and } y \text{ differ in exactly one place}\}$.
- (a) Determine the number of edges in G .
- (b) Prove that G has no odd cycle.
- (c) Prove that G has a perfect matching.
- (d) Determine the maximum size of an independent set in G .
- M10. Let T be a tree with n vertices. For vertices u, v of T , define $d(u, v)$ to be the number of edges in the path from u to v . Let $W(T)$ be the sum of $d(u, v)$ over all $\binom{n}{2}$ pairs of vertices $\{u, v\}$.

- (a) Suppose the tree T is the path on n vertices. Show that

$$W(T) = \frac{1}{2} \sum_{k=1}^{n-1} (k^2 + k).$$

- (b) Now, suppose T is the star graph on n vertices. Show that $W(T) = (n-1)^2$. (N.B. The edge set of the star graph is equal to $\{(u_1, u_i) : 2 \leq i \leq n\}$.)
- (c) Hence, for any tree T , show that

$$(n-1)^2 \leq W(T) \leq \frac{1}{2} \sum_{k=1}^{n-1} (k^2 + k).$$

- M11. (a) Show that, given $2^n + 1$ points with integer coordinates in \mathbb{R}^n , there exists a pair of points among them such that all the coordinates of the midpoint of the line segment joining them are integers.
- (b) Find the number of functions from the set $\{1, 2, 3, 4, 5\}$ onto the set $\{1, 2, 3\}$.

M12. Consider the following LP:

$$\begin{aligned}
 P : & \text{ minimize } x_1 + x_3 \\
 & \text{ subject to} \\
 & \quad x_1 + 2x_2 \leq 5, \\
 & \quad x_2 + 2x_3 = 6, \\
 & \quad x_1, x_2, x_3 \geq 0.
 \end{aligned}$$

- (a) Write down the dual D of P and find the optimal solution of D graphically.
- (b) Using the optimal solution of D , find the optimal solution of P .
- M13. (a) A set S contains integers 1 and 2. S also contains all integers of the form $3x + y$ where x and y are distinct elements of S , and every element of S other than 1 and 2 can be obtained as above. What is S ? Justify your answer.
- (b) Let $\phi(n)$ denote the number of positive integers m relatively prime to n ; $m < n$. Let $n = pq$ where p and q are prime numbers. Then show that

$$\phi(n) = (p-1)(q-1) = pq\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{p}\right)$$

M14. Consider the $n \times n$ matrix $A = ((a_{ij}))$ with $a_{ij} = 1$ for $i < j$ and $a_{ij} = 0$ for $i \geq j$. Let

$$V = \{f(A) : f \text{ is a polynomial with real coefficients}\}.$$

Note that V is a vector space with usual operations. Find the dimension of V , when

- (a) $n = 3$,
 (b) $n = 4$.

Justify your answer.

(iv) STATISTICS

- S1. (a) Let $\{X_n\}_{n \geq 1}$ be a sequence of random variables satisfying $X_{n+1} = X_n + Z_n$ (addition is modulo 5), where $\{Z_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables with common distribution
 $P(Z_n = 0) = 1/2, P(Z_n = -1) = P(Z_n = +1) = 1/4$.
Assume that X_1 is a constant belonging to $\{0, 1, 2, 3, 4\}$. What happens to the distribution of X_n as $n \rightarrow \infty$?
- (b) Let $\{Y_n\}_{n \geq 1}$ be a sequence of independent and identically distributed random variables with a common uniform distribution on $\{1, 2, \dots, m\}$. Define a sequence of random variables $\{X_n\}_{n \geq 1}$ as $X_{n+1} = \text{MAX}\{X_n, Y_n\}$ where X_1 is a constant belonging to $\{1, 2, \dots, m\}$. Show that $\{X_n\}_{n \geq 1}$ is a Markov chain and classify its states.
- S2. Let there be r red balls and b black balls in a box. One ball is removed at random from the box. In the next stage $(a + 1)$ balls of the colour same as that of the removed ball were put into the box ($a \geq 1$). This process was repeated n times. Let X_n denote the total number of red balls at the n -th instant.
- (a) Compute $E(X_n)$.
- (b) Show that if $(r + b)$ is much larger than a and n ,

$$\frac{1}{r}E(X_n) = \left(1 + \frac{na}{r+b}\right) + O\left(\frac{1}{r+b}\right).$$

- S3. Let x_1, x_2, \dots, x_n be a random sample of size n from the gamma distribution with density function

$$f(x, \theta) = \frac{\theta^k}{\Gamma(k)} e^{-\theta x} x^{k-1}, \quad 0 < x < \infty,$$

where $\theta > 0$ is unknown and $k > 0$ is known. Find a minimum variance unbiased estimator for $\frac{1}{\theta}$.

- S4. Let $0 < p < 1$ and $b > 0$. Toss a coin once where the probability of occurrence of head is p . If head appears, then n independent and

identically distributed observations are generated from Uniform $(0, b)$ distribution. If the outcome is tail, then n independent and identically distributed observations are generated from Uniform $(2b, 3b)$ distribution. Suppose you are given these n observations X_1, \dots, X_n , but not the outcome of the toss. Find the maximum likelihood estimator of b based on X_1, \dots, X_n . What happens to the estimator as n goes to ∞ ?

- S5. Let X_1, X_2, \dots be independent and identically distributed random variables with common density function f . Define the random variable N as

$$N = n, \text{ if } X_1 \geq X_2 \geq \dots \geq X_{n-1} < X_n; \text{ for } n = 2, 3, 4, \dots$$

Find $Prob(N = n)$. Find the mean and variance of N .

- S6. (a) Let X and Y be two random variables such that

$$\begin{pmatrix} \log X \\ \log Y \end{pmatrix} \sim N(\mu, \Sigma).$$

Find a formula for $\phi(t, r) = E(X^t Y^r)$, where t and r are real numbers, and E denotes the expectation.

- (b) Consider the linear model $y_{n \times 1} = A_{n \times p} \beta_{p \times 1} + \epsilon_{n \times 1}$ and the usual Gauss-Markov set up where $E(\epsilon) = 0$ and $D(\epsilon) = \sigma^2 I_{n \times n}$, E denotes the *Expectation* and D denotes the *dispersion*.

Assume that A has full rank. Show that $Var(\beta_1^{LS}) = (\alpha - \Gamma^T B^{-1} \Gamma)^{-1} \sigma^2$ where

$$A^T A = \begin{bmatrix} \alpha_{1 \times 1} & \Gamma_{1 \times p-1}^T \\ \Gamma_{1 \times p-1} & B_{p-1 \times p-1} \end{bmatrix}$$

and β_1^{LS} = the least square estimate of β_1 , the first component of the vector β , Var denotes the variance and T denotes transpose.

- S7. Let $p_1(x)$ and $p_2(x)$ denote the probability density functions for classes 1 and 2 respectively. Let P and $(1 - P)$ be the prior probabilities of the classes 1 and 2, respectively. Consider

$$p_1(x) = \begin{cases} x, & x \in [0, 1); \\ 2 - x, & x \in [1, 2]; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$p_2(x) = \begin{cases} x - 1, & x \in [1, 2); \\ 3 - x, & x \in [2, 3]; \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the optimal Bayes risk for this classification problem.
- (b) For which values of P , is the above risk
 - (I) minimized?
 - (II) maximized ?

S8. Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two independent and identically distributed multivariate random vectors with mean $\mathbf{0}$ and covariance matrix $\sigma^2 \mathbf{I}_n$, where $\sigma^2 > 0$ and \mathbf{I}_n is the $n \times n$ identity matrix.

- (a) Show that $\mathbf{X}^T \mathbf{Y} / (\|\mathbf{X}\| \cdot \|\mathbf{Y}\|)$ and $V = \sum (X_i^2 + Y_i^2)$ are independent.
(Here, $\|(a_1, \dots, a_n)\| = \sqrt{a_1^2 + \dots + a_n^2}$).
- (b) Obtain the probability density of $(\sum_{i=1}^n X_i^2 / \sum_{i=1}^n Y_i^2)$.

S9. Let X_1, X_2, \dots, X_n be independent random variables. Let $E(X_j) = j\theta$ and $V(X_j) = j^3\sigma^2$, $j = 1, 2, \dots, n$, $-\infty < \theta < \infty$ and $\sigma^2 > 0$. Here $E(X)$ denotes the expectation and $V(X)$ denotes the variance of the random variable X . It is assumed that θ and σ^2 are unknown.

- (a) Find the best linear unbiased estimator for θ .
- (b) Find the uniformly minimum variance unbiased estimate for θ under the assumption that X_i 's are normally distributed; $1 \leq i \leq n$.

S10. A hardware store wishes to order Christmas tree lights for sale during Christmas season. On the basis of past experience, they feel that the demand v for lights can be approximately described by the probability density function $f(v)$. On each light ordered and sold they make a profit of a cents, and on each light ordered but not sold they sustain a loss of b cents. Show that the number of lights they should order to maximize the expected profit is given by x , which is the solution of the equation:

$$\int_0^x f(v)dv = \frac{a}{a+b}.$$

- S11. Let (X, Y) follow the bivariate normal distribution. Let *mean* of $X = \text{mean}$ of $Y = 0$. Let variance of $X = \text{variance}$ of $Y = 1$, and the correlation coefficient between X and Y be ρ . Find the correlation coefficient between X^3 and Y^3 .
- S12. Let X have probability density function $f(x)(-\infty < x < \infty)$, and we have two hypotheses $H_0 : f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ against $H_1 : f(x) = (1/2) \exp(-|x|)$. Derive the most powerful test at level $\alpha = 0.05$.
- S13. Let X_1, X_2, \dots, X_n be independent and identically distributed observations with a common exponential distribution with mean μ . Show that there is no uniformly most powerful test for testing $H_0 : \mu = 1$ against $H_A : \mu \neq 1$ at a given level $0 < \alpha < 1$ but there exists a uniformly most powerful unbiased test, and derive that test.
- S14. (a) An unbiased die is rolled once. Let the score be $N \in \{1, 2, \dots, 6\}$. The die is then rolled N times. Let X be the maximum of these N scores. Find the probability of the event $(X = 6)$.
- (b) The unit interval $(0,1)$ is divided into two sub-intervals by picking a point at random from the interval. Let Y and Z be the lengths of the longer and shorter sub-intervals, respectively. Find the distribution of Z and show that $\frac{Y}{Z}$ does not have a finite expectation.
- S15. Let X_1, X_2, X_3 be independent and identically distributed observations with a common double exponential distribution with density

$$f(x, \theta) = \frac{1}{2} \exp(-|x - \theta|), \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Suppose that the observations are all distinct.

- (a) Find the maximum likelihood estimator of θ . Give a complete argument for your answer.
- (b) Suppose it is known that $-10 \leq \theta \leq 10$. Find the maximum likelihood estimator of θ . Justify your answer.
- S16. Consider the following linear model

$$y_{ij} = \alpha_i + \beta_j + e_{ij}, \quad i = 1, 2; \quad j = 1, 2, 3;$$

- (a) What is the rank of the error-space? Justify your answer.
- (b) Write down any linear function of observations that belongs to (i) estimation space, (ii) error space.
- (c) Write down a parametric function that is not estimable. Justify your answer.

S17. Let $A = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ be the space obtained by tossing a coin three times. Let $f : A \rightarrow (0, \infty)$ and $x_1 \in A$. For any $x_i \in A$, x_{i+1} is found in the following way.

Toss a fair coin three times and let the outcome be z .

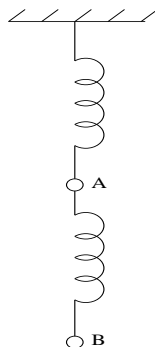
If $f(z) \geq f(x_i)$ then $x_{i+1} = z$, otherwise $x_{i+1} = x_i$.

What can you say about $\lim_{x \rightarrow 0} f(x_i)$? Justify your answer.

S18. Let there be two classes C_1 and C_2 . Let the density function for class C_i be p_i for $i = 1, 2$ where $p_i(x) = ie^{-ix}$; $x > 0$, $i = 1, 2$. Let the prior probability for C_1 be 0.4 and the prior probability for C_2 be 0.6. Find the decision rule for classification of an observation, which provides the minimum probability of misclassification and find its value for that decision rule.

(iii) PHYSICS

- P1. (a) Two particles A and B of equal mass m are attached with two identical massless springs of stiffness constant k in a manner shown in the figure below. When set in longitudinal vibration, find the frequencies and the ratio of the amplitudes, in the normal modes, of the two particles.



- (b) A uniform string fixed at both ends is struck at its centre so as to obtain an initial velocity which varies from zero at the ends to v_0 at the centre. Find the equation of motion of the resulting vibration.
- (c) Show that the zero temperature spin susceptibility of a non-interacting electron gas is

$$\chi = \mu_B g(E_F)$$

where $\mu_B =$ Bohr magneton, $g(E_F) =$ density of states per unit energy at the Fermi energy E_F .

- P2. (a) In the electron spin-orbit interaction the two possible values of j are $l + \frac{1}{2}$ and $l - \frac{1}{2}$. Show that the expectation values of S_Z in the states $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ are $+\frac{m_j \hbar}{2l+1}$ and $-\frac{m_j \hbar}{2l+1}$ respectively. [All the symbols have their usual meanings.]
- (b) A particle in the infinite square well has the initial wave function

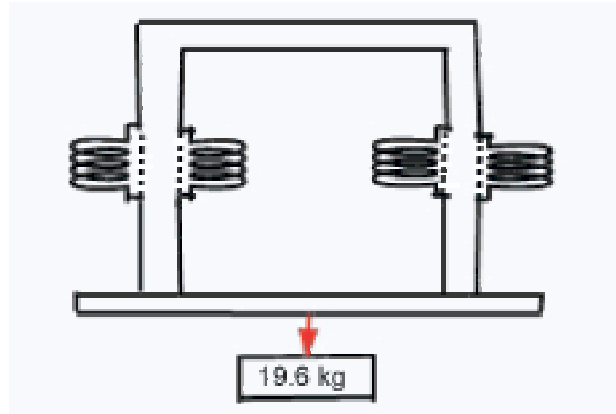
$$\Psi(x, 0) = A \sin^3 \frac{\pi x}{a}, (0 \leq x \leq a)$$

- (i) Determine A .
- (ii) Find $\Psi(x, t)$.
- (iii) Calculate $\langle x \rangle$ as a function of time.
- (c) Consider the operator function

$$\rho(a, a^+) = (1 - e^{-\lambda})e^{-\lambda a^+ a},$$

where a, a^+ are the annihilation and creation operators of the field respectively. Prove that ρ is a density operator.

- P3. A horse-shoe magnet is formed out of a bar of wrought iron of 50 cm length having cross section 6.28 cm^2 . An exciting coil of 500 turns is placed on each limb and connected in series. Find the exciting current necessary for the magnet to lift a load of 19.6 kg (see the figure given below) assuming that the load has negligible reluctance and makes close contact with the magnet. Relative permeability of iron is 700.



- P4. (a) Consider a conducting electron gas at the absolute zero temperature in a weak magnetic field B . The concentrations of spin up and spin down electrons may be parameterised respectively as

$$N_+ = (1/2)N(1 + x), \quad N_- = (1/2)N(1 - x)$$

where N is the total number of electrons.

Evaluate the factor x and calculate the total energy of gas.

- (b) Suppose that there are N spinless particles satisfying Bose-Einstein statistics. The density of available states between E and $E + dE$ is $g(E)$, where

$$g(E) = \begin{cases} 0, & E < 0; \\ \frac{N_0}{E_0}, & E > 0, \end{cases}$$

and N_0 is the number of particles at energy E_0 .

- (i) Find the chemical potential μ as a function of N, E_0, N_0 and β .
- (ii) Can this system have Bose Einstein condensation? Justify your answer.

- P5. (a) Three particles A, B and C of equal mass m are placed on a smooth horizontal plane. A is joined to B and C by light threads AB and AC respectively and $\angle BAC = 60^\circ$. An impulse I is applied to A in the direction BA. Find the initial velocities (immediately after the application of I) of the particles and show that A begins to move in a direction making an angle $\tan^{-1} \frac{\sqrt{3}}{7}$ with BA.
- (b) A particle on a frictionless horizontal plane at a latitude λ is given an initial speed u in the northern direction. Prove that it describes a circle of radius $\frac{u}{2\omega \sin \lambda}$ with a period $T = \frac{\pi}{\omega \sin \lambda}$ where ω is the angular velocity of the earth.
- P6. Two electrons are confined in a one dimensional box of length a . A clever experimentalist has made arrangements such that both the electrons are in the same spin state. Ignore the Coulomb interaction between the electrons.
- (a) Write down the ground state wave function of the two-electron system.
- (b) What is the probability that both the electrons are found on the same half of the box?
- (c) Will the nature of construction of the wave function in (a) hold if Coulomb interaction is included? Give reasons for your answer.
- (d) In the above problem, consider two charged π -mesons instead of two electrons. Write down the ground state wave function ignoring the Coulomb interactions.
- P7. (a) Consider an eigenstate of a two-particle system, in one dimension, represented by the wave function

$$\psi(x_1, x_2) = e^{iP(x_1+x_2)/(2\hbar)} e^{-(Mk/2)^{\frac{1}{2}}(x_1-x_2)^2/(2\hbar)}.$$

Here x_1 and x_2 are the positions of the two particles of equal mass (M) moving in one dimension and interacting with a harmonic oscillator force $F = -k(x_1 - x_2)$.

- (i) Calculate the total energy associated with the relative motion.
- (ii) Also calculate the mean absolute value of the relative momentum.

- (b) A particle of mass m_1 MeV/c² and kinetic energy k_1 MeV collides with a stationary particle of mass m_2 MeV/c². After the collision, the two particles stick together. Calculate
- the initial momentum of the two-particle system, and
 - the final velocity of the two-particle system.
- P8. (a) A monatomic classical gas (made up of N atoms, each having mass m) is contained in a cylinder with cross sectional area A and height h . The system is subject to a linear potential $U(z) = bz$, where z is the vertical coordinate. Assume that the temperature T is uniform in the cylinder. Find the free energy of this system.
- (b) Consider a sequence of $2N$ ions of alternating charges $\pm q$ arranged on a line with a repulsive potential $\frac{A}{R^n}$ between nearest neighbours in addition to the usual Coulomb potential. Find the equilibrium separation R_0 and the equilibrium energy. Also evaluate the nearest neighbour distance when the potential energy is zero. (Neglect the surface effect).
- P9. (a) Consider an electromagnetic wave in free space of the form,

$$\vec{E}(x, y, z, t) = (E_x^0(x, y)\hat{i} + E_y^0(x, y)\hat{j})e^{i(kz - \omega t)},$$

$$\vec{B}(x, y, z, t) = (B_x^0(x, y)\hat{i} + B_y^0(x, y)\hat{j})e^{i(kz - \omega t)}.$$

Here \vec{E}^0 and \vec{B}^0 are in the xy plane.

Show that \vec{E}^0 and \vec{B}^0 satisfy the time independent Maxwell's equations.

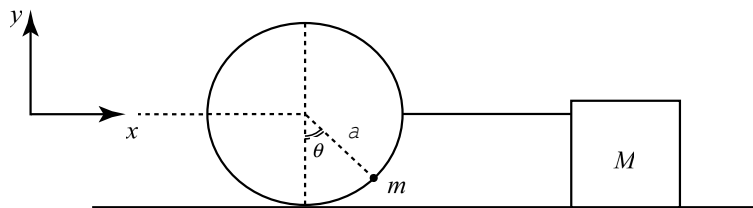
- (b) Two point charges of magnitude e are located at the end points of a line of length $2l$ in the xy plane with its midpoint passing through the origin. The line is rotating about the z -axis in the anti-clockwise direction with a constant angular velocity ω . Calculate the following:
- electric dipole moment of the system.
 - magnetic dipole moment of the system.
- (c) A parallel plate capacitor (having perfectly conducting plates) with plate separation d is filled with layers of two different materials. The first layer has dielectric constant ϵ_1 and conductivity σ_1 ; the second layer has dielectric constant ϵ_2 and conductivity σ_2 . Their thicknesses are d_1 and d_2 , respectively. A potential

difference of V is applied across the capacitor. Neglecting the edge effect,

- (i) calculate the electric field in each of the two dielectric media.
- (ii) what is the current flowing through the capacitor?
- (iii) what is the total surface charge density on the interface between the two layers?

P10. (a) Suppose a planet is moving in a circular orbit of radius R . It is stopped suddenly in its orbit. Show that it would fall onto the sun in a time which is $\frac{\sqrt{2}}{8}$ times of its period.

(b) A block of mass M is rigidly connected to a massless circular track of radius a fixed in a vertical plane on a horizontal table as shown in the figure. A particle of mass m is confined to move without friction on the circular track (in the vertical plane).



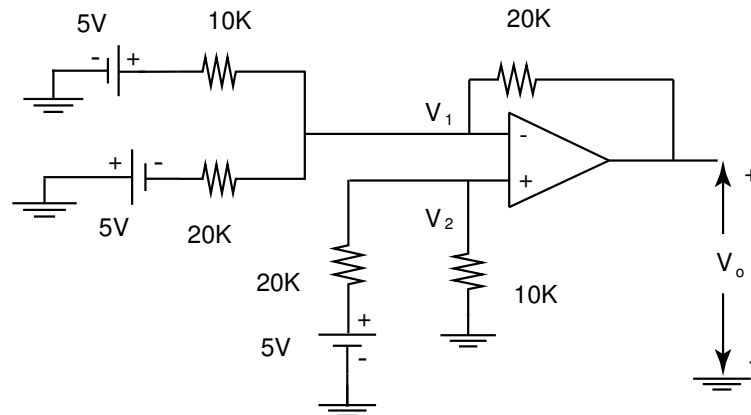
- (i) Set up the Lagrangian using x and θ as the coordinates.
- (ii) Find the equations of motion.
- (iii) Solve the equation of motion for θ (for small θ).

P11. For an intrinsic semiconductor with a gap width of 1 eV, calculate the position of Fermi level at $T = 0^0K$ and $T = 300^0K$, if $m_h = m_e$, where m_e and m_h are effective masses of an electron and a hole respectively. Also calculate the density of free electrons and holes at $T = 300^0K$ and $T = 600^0K$, given that $\log_{10}e = 0.40$, $\left(\frac{2\pi\sqrt{m_e m_h} kT}{h^2}\right)^{\frac{3}{2}} = 0.5 \times 10^{19}/cc$. If the above semiconductor is now doped with a group V element with a doping concentration of $10^{14}/cc$, then compute the electron and hole densities of the doped semiconductor specimen.

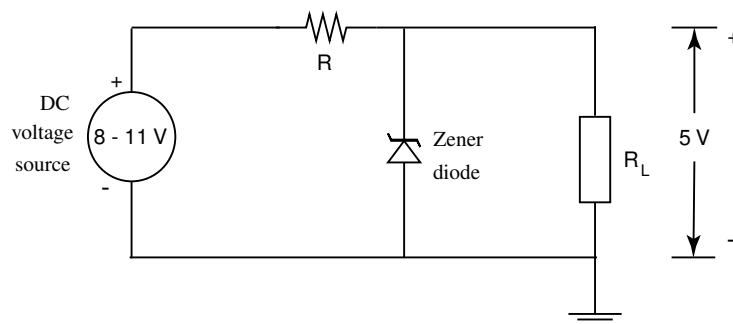
P12. (a) A negative feedback amplifier has a voltage gain of 100. Variations of the voltage gain up to $\pm 2\%$ can be tolerated for some specific application. If the open-loop gain variations of $\pm 10\%$ are

expected owing to production spreads in device characteristics, determine the minimum value of the feedback ratio β and also the open loop gain to satisfy the above specification.

(b) Calculate the output voltage V_0 for the following network:

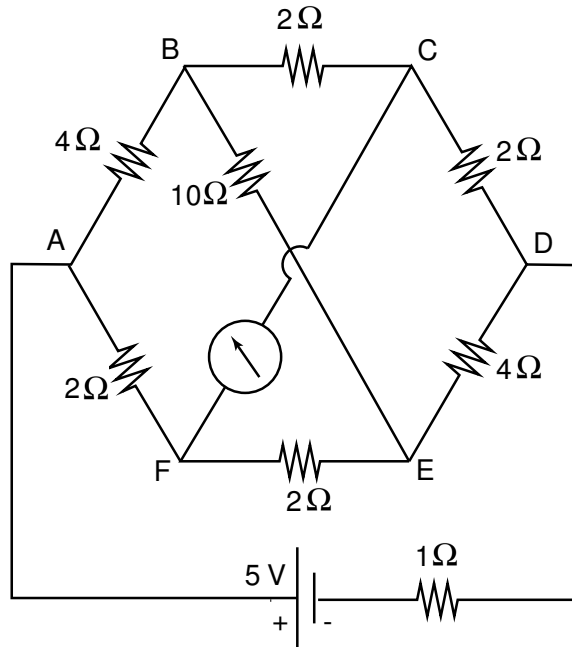


P13. (a) Consider the following circuit for deriving a +5 volt power supply to a resistive load R_L from an input d-c voltage source whose voltage may vary from 8V to 11V. The load R_L may draw a maximum power of 250 mW. The Zener diode has a breakdown voltage of 5 volts.



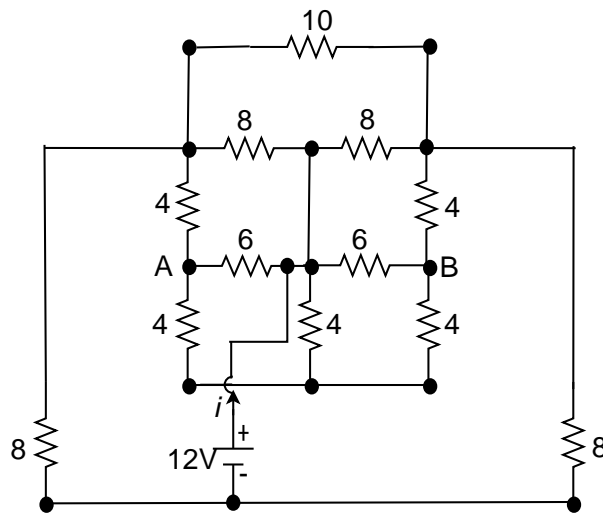
Compute the maximum value of the resistance R and also the power dissipation requirements for R and the Zener diode. Assume that the minimum breakdown current of the Zener diode is negligible compared to the load current.

- (b) Consider the following circuit. Calculate the potential difference between the points F and C , as shown by the ideal voltmeter.

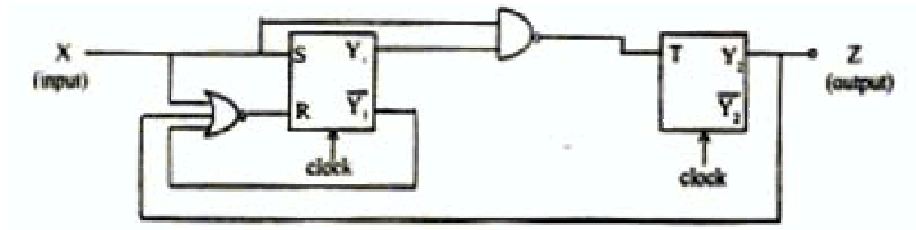


(iv) RADIOPHYSICS/TELECOMMUNICATIONS/ELECTRONICS/
ELECTRICAL ENGINEERING

- R1. Design a sequential machine that produces an output 1 whenever a substring of 5 consecutive symbols in the input starts with two 1's and contains exactly three 1's. If a substring of 5 symbols starts with two 1's, the analysis of the next substring does not begin until the processing of the current substring is complete. Realize this circuit with the minimum number of NAND gates and flip flops.
- R2. Consider the following circuit, where all the resistances are in *ohms*. Calculate the potential difference between A and B . Also compute the current i drawn from the battery.



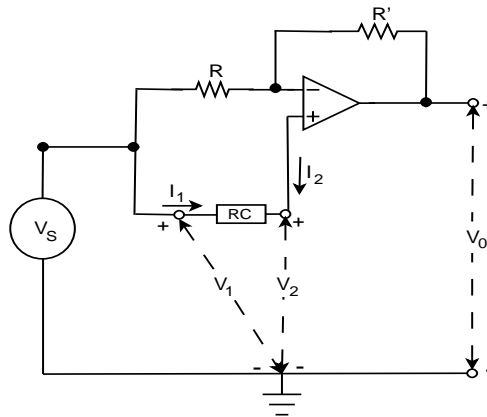
R3. Draw the state table for the synchronous sequential circuit shown in the figure below.



R4. Consider the following circuit of an ideal OP-AMP and an RC two-port network. Assume that the RC two-port network is represented in terms of its y parameters, i.e., $y_{11} = \frac{I_1}{V_1} |V_2=0$, $y_{12} = \frac{I_1}{V_2} |V_1=0$, $y_{21} = \frac{I_2}{V_1} |V_2=0$, and $y_{22} = \frac{I_2}{V_2} |V_1=0$. Show that the voltage gain of the above circuit is given by

$$\frac{V_o}{V_s} = -\frac{y_{21}(1+k) + ky_{22}}{y_{22}},$$

where $k = \frac{R'}{R}$.



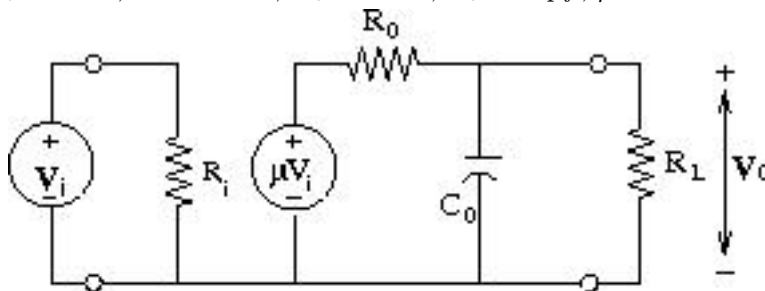
R5. Consider a voltage amplifier circuit shown in the figure below, where R_i and R_o represent the input and output impedances respectively, C_o is the total parasitic capacitance across the output port, μ is the amplifier gain and the output is terminated by a load resistance R_L .

(a) Calculate the current, voltage and power gain in decibels (dB) of the amplifier, when

$$R_i = 1M\Omega, R_L = 600\Omega, R_o = 100M\Omega, C_o = 10pf, \mu = 10.$$

(b) Calculate the 3-dB cutoff frequency of the amplifier when

$$R_i = 5K\Omega, R_L = 1K\Omega, R_o = 100\Omega, C_o = 10pf, \mu = 2.$$



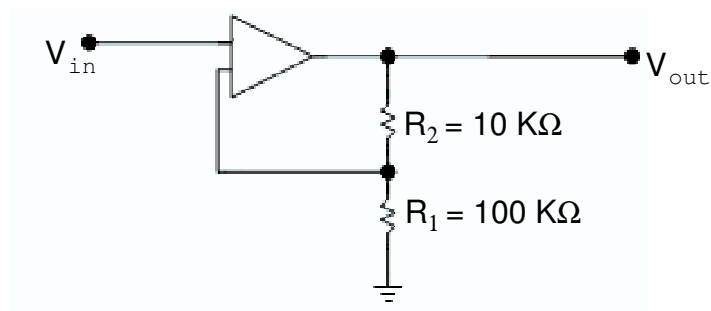
R6. A 50 HP (37.3 KW), 460 V DC shunt motor running freely takes a current of 4 A and runs at a speed of 660 rpm. The resistance of the armature circuit (including brushes) is 0.3 ohm and that of the shunt field circuit is 270 ohm.

(a) Determine (i) the total current, and (ii) the speed of the motor when it is running at full load.

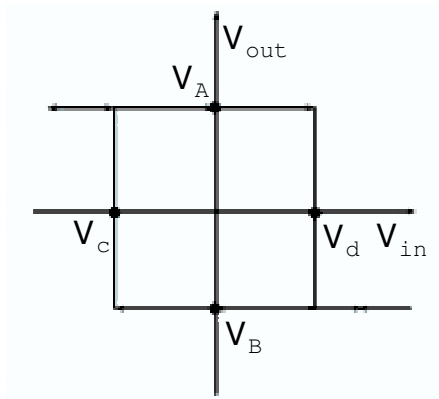
(b) Determine the armature current at which the efficiency is maximum (ignore the effect of armature reaction).

- R7. (a) Three 100 ohm, non-inductive resistances are connected in (i) Star and (ii) Delta configurations across 400 V, 50 Hz, 3-phase main. Calculate the power taken from the supply system in each case.
- (b) In the event of one of the three resistances getting open circuited, what variation would be in the value of the total power taken in each of the two configurations?

R8. Consider the following circuit with an OP-AMP.



The plot of output voltage V_{out} vs. input voltage V_{in} for the given circuit is as follows.



Let $V_A = 10V$ and $V_B = -10V$. Assume that $V_{in} < V_c$, and is gradually increasing. The output voltage $V_{out} = V_A$ until $V_{in} = V_d$ and then falls to V_B . The output remains at V_B for $V_{in} > V_d$. Similarly, if V_{in} is initially $> V_d$ and gradually reduced, V_{out} remains at V_B until $V_{in} = V_c$, and then rises to V_A for all values $V_{in} < V_c$.

- (a) Explain why the circuit behaves in this fashion, and
 (b) calculate the values of V_c and V_d .
- R9. Assume that an analog voice signal which occupies a band from 300 Hz to 3400 Hz, is to be transmitted over a Pulse Code Modulation (PCM) system. The signal is sampled at a rate of 8000 samples/sec. Each sample value is represented by 7 information bits plus 1 parity bit. Finally, the digital signal is passed through a raised cosine roll-off filter with the roll-off factor of 0.25. Determine
- (a) whether the analog signal can be exactly recovered from the digital signal;
 (b) the bit duration and the bit rate of the PCM signal before filtering;
 (c) the bandwidth of the digital signal before and after filtering;
 (d) the signal to noise ratio at the receiver end (assume that the probability of bit error in the recovered PCM signal is zero).
- R10. A logic circuit is to be designed having four inputs x_1, x_0, y_1 and y_0 and the three outputs z_1, z_2 and z_3 . The pair of bits x_1x_0 and y_1y_0 represent two binary numbers X and Y with x_1 and y_1 as the most significant bits. z_1 is 1 if X is larger than Y and z_2z_3 represent the difference between the two numbers X and Y . Find the minimum sum of product expressions for z_1, z_2 and z_3 .
- R11. A message $bbccfe\backslash$ needs to be encoded using arithmetic coding. The probabilities of message symbols are shown in the following table.

| | | | | | | | |
|-------------|------|-----|-----|------|-----|-----|--------------|
| symbol | a | b | c | d | e | f | \backslash |
| Probability | 0.05 | 0.2 | 0.1 | 0.05 | 0.3 | 0.2 | 0.1 |

Using the symbol probabilities shown in the above table, find

- (a) a fractional value that is to be transmitted after encoding the message $bbccfe\backslash$,
 (b) the exact decoding scheme of the message from the fractional value estimated at the encoding stage, and
 (c) the number of bits required to represent the encoded message after arithmetic coding.

R12. Consider two identical parallel plate air capacitors in series. The combination is maintained at the constant potential difference of 35 volts. Now a dielectric sheet of dielectric constant 4 and thickness equal to 0.8 of the air gap is inserted into one of the capacitors, so that it spans the whole area of the plates of the capacitors. Calculate the voltage across this capacitor and the ratio of electrostatic energies stored in the two capacitors.

R13. Two linear, time-invariant (LTI) discrete-time systems with frequency responses as indicated below, are cascaded to form another LTI system.

$$H_1(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| < \pi/2, \\ 0 & \text{otherwise.} \end{cases}$$

$$H_2(e^{j\omega}) = \begin{cases} -j & \text{if } 0 < \omega < \pi, \\ j & \text{if } -\pi < \omega < 0. \end{cases}$$

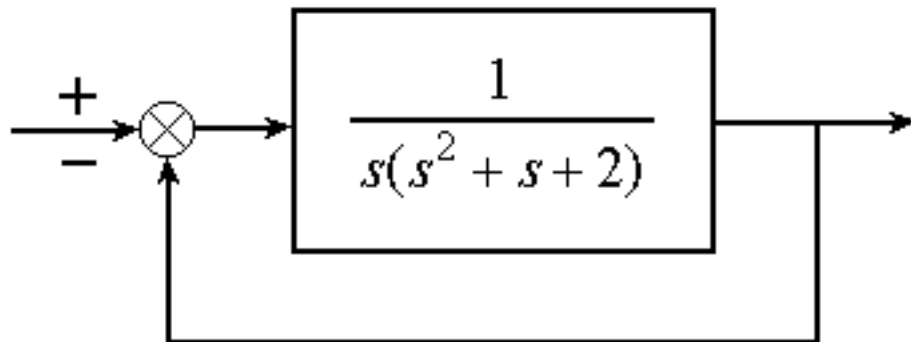
- (a) Determine the overall frequency response.
 (b) A continuous time signal $x(t)$ bandlimited to 1kHz. has a Fourier Transform as follows:

$$X(j\Omega) = \begin{cases} 1 - \Omega/2\pi 1000 & \text{if } |\Omega| < 2\pi 1000, \\ 0 & \text{otherwise.} \end{cases}$$

It is sampled at a rate of 2kHz. to obtain a discrete-time signal $x[n]$. This passes through the cascaded system mentioned earlier to produce a discrete-time signal $y[n]$. Draw $|X(e^{j\omega})|$ and $|Y(e^{j\omega})|$.

- (c) Is it possible to recover $x[n]$ from $y[n]$? Explain.

R14. Determine the stability of the following closed loop system. S represents Laplace operator.



(v) COMPUTER SCIENCE

- C1. (a) Write the smallest real number greater than 6.25 that can be represented in the IEEE-754 single precision format (32-bit word with 1 sign bit and 8-bit exponent).
- (b) Convert the sign-magnitude number **10011011** into a 16-bit 2's complement binary number.
- (c) The CPU of a machine is requesting the following sequence of memory accesses given as word addresses: 1, 4, 8, 5, 20, 17, 19, 56. Assuming a direct-mapped cache with 8 one-word blocks, that is initially empty, trace the content of the cache for the above sequence.
- C2. Consider a collection of n binary strings S_1, \dots, S_n . Each S_i is of length l_i bits where $1 \leq l_i \leq k$.
- (a) Write a function **prefix(S,T)** in C programming language that takes two binary strings S, T and returns 1 if S is a prefix of T , else it returns 0. For example, **prefix(001,00101)** returns 1 but **prefix(010,00101)** returns 0.
- (b) Suppose we want to report all the pairs (i, j) for which S_i is a prefix of S_j , ($1 \leq i \neq j \leq n$). How many times do we need to call the **prefix** function described above?
- (c) Present an $O(nk)$ time algorithm to report all the (i, j) 's as mentioned in (b). (Hint: Use a binary tree with each edge marked as 0 or 1; a path from the root to a node in the tree represents a binary string.)

- C3. (a) Write a computer program in the C language that takes an array A of $2n$ distinct floating point numbers, and prints the maximum and the minimum values of the array A , along with their indices. *(Full credit will be given only if your program does not make more than $(3n - 2)$ floating point comparisons.)*
- (b) Briefly argue that your program indeed computes the maximum and the minimum values correctly.
- C4. Let $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} + 1$ for $n > 2$.
- (a) Express 63 as a sum of distinct a_i 's.
- (b) Write an algorithm to express any positive integer k as a sum of at most $\lceil \log_2 k \rceil$ many distinct a_i 's.
- (c) Prove the correctness of your algorithm.
- C5. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of n integers. A pair (x_i, x_j) is said to be the closest pair if $|x_i - x_j| \leq |x'_i - x'_j|$, for all possible pairs (x'_i, x'_j) , $i', j' = 1, 2, \dots, n, i' \neq j'$.
- (a) Describe a method for finding the closest pair among the set of integers in S using $O(n \log_2 n)$ comparisons.
- (b) Now suggest an appropriate data structure for storing the elements in S such that if a new element is inserted to the set S or an already existing element is deleted from the set S , the current closest pair can be reported in $O(\log_2 n)$ time.
- (c) Briefly explain the method of computing the current closest pair, and necessary modification of the data structure after each update. Justify the time complexity.
- C6. Let A be an $n \times n$ matrix such that for every 2×2 sub-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of A , if $a < b$ then $c \leq d$. Note that for every pair of rows i and j , if a_{ik} and a_{jl} are the largest elements in i -th and j -th rows of A , respectively, then $k \leq l$ (as illustrated in the 5×5 matrix below).

$$\begin{bmatrix} 3 & 4 & 2 & 1 & 1 \\ 7 & 8 & 5 & 6 & 4 \\ 2 & 3 & 6 & 6 & 5 \\ 5 & 6 & 9 & 10 & 7 \\ 4 & 5 & 5 & 6 & 8 \end{bmatrix}$$

- (a) Write an algorithm for finding the maximum element in each row of the matrix with time complexity $O(n \log n)$.
- (b) Establish its correctness, and justify the time complexity of the proposed algorithm.

C7. Consider a file consisting of 100 blocks. Assume that each disk I/O operation accesses a complete block of the disk at a time. How many disk I/O operations are involved with contiguous and linked allocation strategies, if one block is

- (a) added at the beginning?
- (b) added at the middle?
- (c) removed from the beginning?
- (d) removed from the middle?

C8. (a) Consider the context-free grammar $G = (\{S, A\}, \{a, b\}, S, P)$, where

$$P = \{S \rightarrow AS, \\ S \rightarrow b, \\ A \rightarrow SA, \\ A \rightarrow a\}$$

Show that G is left-recursive. Write an equivalent grammar G' free of left-recursion.

(b) Consider the grammar $G = (\{S, T\}, \{a, \pi, (,), +\}, S, P)$, where

$$P = \{S \rightarrow a|\pi|(T), \\ T \rightarrow T + S|S\}$$

Find the parse tree for the sentence:

$$(((a + a) + \pi + (a)) + a)$$

C9. (a) Five batch jobs P_1, \dots, P_5 arrive almost at the same time. They have estimated run times of 10, 6, 2, 4 and 8 ms. Their priorities are 3, 5, 2, 1 and 4 respectively, where 1 indicates the highest priority and 5 indicates the lowest. Determine the average turn-around and waiting time for the following scheduling algorithms:

- (i) Round robin with time quantum of 5 ms,

- (ii) Priority scheduling.
 - (b) The access time of a cache memory is 100 ns and that of main memory is 1000 ns. It is estimated that 80% of the memory requests are for read and the remaining 20% are for write. The hit ratio for read access is 0.9. A write through procedure is used.
 - (i) What is the average access time of the system considering only memory read cycles?
 - (ii) What is the average access time of the system considering both read and write requests?
- C10. (a) A program P consisting of 1000 instructions is run on a machine at 1 GHz clock frequency. The fraction of floating point (FP) instructions is 25%. The average number of clock-cycles per instruction (CPI) for FP operations is 4.0, and that for all other instructions is 1.0.
- (i) Calculate the average CPI for the overall program P .
 - (ii) Compute the execution time needed by P in seconds.
- (b) Consider a 100Mbps token ring network with 10 stations having a ring latency of 50 μ s (the time taken by a token to make one complete rotation around the network when none of the stations is active). A station is allowed to transmit data when it receives the token, and it releases the token immediately after transmission. The maximum allowed holding time for a token (THT) is 200 μ s.
- (i) Express the maximum efficiency of this network when only a single station is active in the network.
 - (ii) Find an upper bound on the token rotation time when all stations are active.
 - (iii) Calculate the maximum throughput rate that one host can achieve in the network.
- C11. Consider a graph G (called an interval graph) whose nodes correspond to a set of intervals on the real line. The i -th interval is denoted by $[\alpha_i, \beta_i]$, where $0 \leq \alpha_i < \beta_i$. An edge between two nodes (i, j) implies that the corresponding intervals $[\alpha_i, \beta_i]$ and $[\alpha_j, \beta_j]$ overlap.
- (a) Consider the set of intervals $[3, 7]$, $[2, 4]$, $[2, 3]$, $[1, 5]$, $[1, 2]$, $[6, 7]$, $[10, 16]$, $[11, 12]$. Draw the corresponding interval graph and identify the largest subgraph where all the nodes are connected to each other.

- (b) Write an algorithm which takes the interval graph G as input and finds the largest subgraph of G in which all the nodes are connected to each other. What is the time complexity of your algorithm?
- (c) Given a list of intervals, write an algorithm to list all the connected components in the corresponding interval graph. What is the time complexity of your algorithm?
- C12. (a) A functional dependency $\alpha \rightarrow \beta$ is called a *partial* dependency if there is a proper subset γ of α such that $\gamma \rightarrow \beta$. Show that every partial dependency is a transitive dependency.
- (b) Let $R = (A, B, C, D, E)$ be a schema with the set $F = \{A \rightarrow BC, CD \rightarrow E, B \rightarrow D, E \rightarrow A\}$ of functional dependencies. Suppose R is decomposed into two schema $R_1 = (A, B, C)$ and $R_2 = (A, D, E)$
- (i) Is this decomposition loss-less? Justify.
- (ii) Is this decomposition dependency preserving? Justify.
- (c) Consider the relations $r_1(A, B, C)$, $r_2(C, D, E)$ and $r_3(E, F)$. Assume that the set of all attributes constitutes the primary keys of these relations, rather than the individual ones. Let $V(C, r_1)$ be 500, $V(C, r_2)$ be 1000, $V(E, r_2)$ be 50, and $V(E, r_3)$ be 150, where $V(X, r)$ denotes the number of distinct values that appear in relation r for attribute X . If r_1 has 1000 tuples, r_2 has 1500 tuples, and r_3 has 750 tuples, then give the ordering of the natural join $r_1 \bowtie r_2 \bowtie r_3$ for its efficient computation. Justify your answer.
- C13. (a) (i) Write a Context Free Grammar (CFG) for structure definitions in C . Assume that the only allowable types are char, int, and float (you need not handle pointers, arrays, structure, fields, etc.).
- (ii) Assume that chars are stored using 1 byte each; ints and floats are stored using 4 bytes each and are aligned at 4 byte boundaries. Add semantic rules to your grammar to calculate the number of bytes required to store the structure defined by your grammar.
- (b) (i) Compute the canonical collection of sets of $LR(1)$ items (*i.e.* canonical LR items) for the following grammar: $S \rightarrow aXcd$,

$S \rightarrow aYce, X \rightarrow b, Y \rightarrow b$. Is the grammar $LR(1)$? Briefly justify.

- (ii) Give an example of a grammar that is unambiguous but not $LR(2)$. Briefly justify/explain your example.

C14. An operating system allocates memory in units of 1 KB pages. The address space of a process can be up to 64 MB in size; however, at any point of time, a process can be allocated at most 16 MB of physical memory. In addition the kernel uses 65 KB of physical memory to store page table entries of the current process. The OS also uses a translation-lookaside buffer (TLB) to cache *page table entries*. You are also given the following information:

- size of a page table entry is 4 bytes,
- TLB hit ratio is 90%,
- time for a TLB lookup is negligible,
- time for a memory read is 100 nanoseconds,
- time to read a page from the swapping device into physical memory is 10 milliseconds.

Calculate the effective memory access time for a process whose address space is 20 MB? Assume that memory accesses are random and distributed uniformly over the entire address space.

- C15. (a) What are the conditions which must be satisfied by a solution to the critical section problem?
- (b) Consider the following solution to the critical section problem for two processes. The two processes, P_0 and P_1 , share the following variables:

```
var flag : array [0..1] of Boolean;
        (* initially false *)
    turn : 0..1;
```

The program below is for process P_i ($i = 0$ or 1) with process P_j ($j = 1$ or 0) being the other one.

```
repeat
    flag[i] <- true;
    while (flag[j])
        do if (turn = j)
```

```

then begin
    flag[i] <- false;
    while (turn = j) do skip;
end;
...
CRITICAL SECTION
...
turn <- j;
flag[i] <- false;
...
REMAINDER SECTION
...
until false;

```

Does this solution satisfy the required conditions?

- C16. (a) Construct an AVL tree of height 5 with minimum number of nodes.
- (b) Consider a B-tree of order 3.
- (i) Trace the insertion of the keys a, g, f, b, k, d, h, m , into an initially empty tree, in lexicographic order.
 - (ii) Sketch the B-tree upon deletion of keys h, d .