Test Code : RP : (Short Answer type) 2008

Junior Research Fellowship in Theoretical Physics and Applied Mathematics

The candidates for Junior Research Fellowships in Applied Mathematics and Theoretical Physics will have to take two tests−Test MIII (objective type) in the forenoon session and Test RP (short answer type) in the after noon session.

The RP test booklet will consist of two parts. The candidates are required to answer from Part I and only one of the remaining parts II, III and IV.

The syllabus and sample questions for the test are as follows.

PART-I

Mathematical and logical reasoning

Syllabus

B.Sc. Pass Mathematics syllabus of Indian Universities.

Sample Questions

1. If $2k+1$ is a perfect square (k is integer, > 3), then show that $k+1$ can be written as a sum of two squares.

2. Let f be a real valued function defined on the interval $[-2, 2]$ as:

$$f(x) \;=\; \begin{cases} (x+1)2^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

   i) Find the range of the function.

   ii) Is f continuous at every point in $(-2, 2)$? Justify your answer.

3. A point moves in the (x, y)-plane with position $(x(t), y(t))$ at time t, where $x(t) = 1/t$ and $y(t) = \frac{1}{\sqrt{2}}(1+t)$ for $t > 0$. Show that the point comes closest to the origin when $t = 1$.

4. The position of a particle moving in a plane is given by $x = \sin \omega t$, $y = \cos \alpha\omega t$. Show that the trajectory repeats itself periodically only if α is a rational number.

5. Evaluate $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

6. X is a uniformly distributed random variable with probability density function

$$f(x) = \begin{cases} \dfrac{5}{a} & \text{for} \quad -\frac{a}{10} \leq x \leq \frac{a}{10} \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

where a is a nonnegative constant. If $P(|x| < 2) = 2P(|x| > 2)$, then find a.

7. Suppose a, b, c are positive integers such that

$$abc + ab + bc + ca + a + b + c = 1000$$

Find the value of $a + b + c$.

8. The rate of cooling of a substance in moving air is proportional to the difference of temperatures of the substance and the air. A substance cools from 36^0C to 34^0C in 15 minutes. Find the time required to cool the substance from 36^0C to 32^0C, considering the constant temperature of air as 30^0C.

9. Assuming nothing else blocks their view, how far can two people, each of hight h (in meters), walk from each other until they can no longer see each other due to earth's curvature. (Take earth to be a perfect sphere with radius R (in meters)).

10. If $f(x) \geq 0$ for $a \leq x \leq b$, we know that $\int_a^b f(x)dx \geq 0$. However, by actual calculation we find that $\int_{-2}^{1} \frac{dx}{x^2} = -\frac{3}{2}$, which is clearly absurd. What went wrong ?

11. Let $f(x, y) = |xy|^{\frac{1}{2}}$. Is this function differentiable at (0,0)?

12. If $f(x, f(y)) = x^p y^q$ all x, y, then show that $p^2 = q$ and find $f(x)$.

13. Show that the area of the triangle formed by z, iz, and $z + iz$ is $\frac{r^2}{2}$ where $r = |z|$ and $z = a + ib$, a, b being real nonzero numbers.

14. If the lines $3x - 4y + 4 = 0$, $6x - 8y - 7 = 0$ are tangents to the same circle, find the radius of the circle.

15. A body of mass m thrown at an angle α to the horizontal plane with an initial velocity v_0, moves under the action of the force of gravity and the force of resistance $\vec{R}$ of the air given by $\vec{R} = \gamma \vec{v}$ where $\vec{v}$ is the velocity vector and γ is a constant. Determine the maximum height attained by the body.

16. A particle of mass m moves under a conservative force with potential energy $V(x) = \frac{Cx}{(a^2 + x^2)}$, where C and a are positive constants. Find the position of stable equilibrium and the equation of motion if there is a small oscillation about the stable position.

17. Find the maximum possible value of xy^2z^3 subject to the conditions x, y, $z \geq 0$ and $x + y + z = 3$.

18. A point moves in the (x, y)-plane such that at time $t (> 0)$ it has the coordinates $(1/t, (1 + t)/\sqrt{2})$. Find when it comes closest to the origin.

PART-II

Applied Mathematics

Syllabus

1. *Abstract algebra* : Groups, rings, fields.

2. *Real analysis* : Functions of single and several variables, metric space, normed linear space, Riemann Integral, Fourier series.

3. *Differential equations* : ODE – Existence of solution, fundamental system of integrals, elementary notions, special functions. PDE upto second order, equations of parabolic, hyperbolic and elliptic type.

4. *Dynamics of particles and rigid bodies* : Motion of a particle in a plane and on a smooth curve under different laws of resistance, kinematics of a rigid body, motion of a solid body on an inclined smooth or rough plane.

5. *Functions of complex variables* : Analytic function, Cauchy's theorem Taylor and Laurent series, singularities, branch-point, contour integration, analytic continuation.

6. *Numerical analysis* : Solution of a system of linear equations, polynomial interpolation, numerical integration formula (Newton-Cotes type).

7. *Fluid Mechanics* : Kinematics of fluid, equation of continuity, irrotational motion, velocity potential, dynamics of ideal fluid, Eulerian and Lagrangian equations of motion, stream function, sources, sinks and doublets, vortex, surface waves, group velocity, Viscous flow – Navier Stokes equation, boundary layer theory, simple problems.

8. *Probability and statistics* : Probability axioms, conditional probability, probability distribution, mathematical expectations, characteristic functions, covariance, correlation coefficient. Law of large numbers, central limit theorem. Random samples, sample characteristics, estimation, statistical hypothesis, Neyman pearson theorem, likelihood ratio testing.

Sample Questions

1. Let m and n be positive integers and F is a field. Let f_i $(i = 1, 2, \ldots, m)$ be linear functionals on F^n. For α in F^n, let $T\alpha = (f_1(\alpha), f_2(\alpha), \ldots \ldots, f_m(\alpha))$. Show that T is a linear transformation from F^n into F^m. Also show that every linear transformation from F^n into F^m is of the above form for some f_i $(i = 1, 2, \ldots, m)$.

2. (a) Let $X_1 = [1, 2]$ and $X_2 = [0, 1]$. Let d_1 denote the Euclidean metric in X_1 and let $d_2(x, y) = 2|x - y|$ in X_2. Show that (X_1, d_1) and (X_2, d_2) are equivalent metric spaces.

b) Two different metrics on the space $X = x \in R : 0 < x \leq 1$ are defined by $d_1(x,y) = |x-y|$ and $d_2(x,y) = |\frac{1}{x} - \frac{1}{y}|$. Are the spaces (X, d_1) and (X, d_2) equivalent? Give reasons for your answer.

3. (a) A particle of mass m moves on the outside of a smooth circle of radius a kept vertically. The particle starts from a point P at an angular distance α from the highest point A and is allowed to slide down. Show that the particle leaves the circle at point Q whose angular distance θ from A is given by $\cos\theta = \frac{2}{3}\cos\alpha$.

   (b) An ice cube is melting, becoming smaller but still cubical.The rate of melting is proportional to the surface area of the cube. Let $v(t)$ be the volume of the cube at time t. Explain why $\frac{dv}{dt} = \alpha v^{\frac{2}{3}}$ for some constant α. Will α be positive or negative.

4. Show that if the solution of the ODE

$$2xy'' + (3-2x)y' + 2y = 0$$

is expressed in the form $y = \sum_{n=0}^{\infty} a_n\, x^{n+\sigma}$, σ can take two possible values. Find the relation between a_n and a_{n+1}, and show that one solution reduces to a polynomial.

5. Let $u(x,y)$ satisfy the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \ y \geq 0$$

with the boundary condition $u(x,0) = f(x), \quad -\infty < x < \infty$ and $u(x,y) \to 0$ as $y \to \infty$; u, $\frac{\partial u}{\partial x} \to 0$ as $|x| \to \infty$. Use Fourier transform technique to show that

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{y^2 + (x-t)^2}dt.$$

6. Reduce the partial differential equation

$$(a-1)^2\frac{\partial^2 u}{\partial x^2} - y^{2a}\frac{\partial^2 u}{\partial y^2} = ay^{a-1}\frac{\partial u}{\partial y}$$

to canonical form and find its general solution.

7. a) Show that

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) \quad \text{for} \quad 0 \leq x \leq \pi.$$

Deduce that
$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

b) If $f(z)$ $(z = x+iy)$ is an analytic function, prove that $\log|f(z)|$ is harmonic.

8. Let $f(x)$ be defined on $[1,\infty)$. Also $f(x)$ is continuous and differentiable on that interval and further the derivative $f'(x)$ is given by $f'(x)=\dfrac{1}{x^2+(f(x))^2}$. Show that for all $x\geq 1$, $f(x)\leq 1+\dfrac{\pi}{4}$.

9. Examine the extremum values from
$$x^3+3xy^2-15x^2-15y^2+72x \;=\; 0.$$

10. Calculate the eigenvalues and eigenvectors of the matrix $A=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$. Show that the eigenvectors are orthogonal and form a basis in C^2. Find also a unitary matrix U.

11. Find the integral surface of the differential equation
$$(x-y)p+(y-x-z)q=z,$$
through $z=1$, $x^2+y^2=1$.

12. Evaluate the integral $\displaystyle\int_c \frac{e^{1/z^2}}{z^2+1}dz$, where c is the curve given by $|z-i|=\dfrac{7}{2}$, the integration being taken counterclockwise.

13. Construct an integration rule of the form
$$\int_{-1}^{1} f(x)dx \simeq c_0 f\left(-\frac{1}{2}\right)+c_1 f(0)+c_2 f(\frac{1}{2})$$
which is exact for all polynomials of degree ≤ 2. Find the error term.

14. Verify that $\left(\omega^2 tx, -\omega^2 ty+U\omega^3 t^3\right)$ is a possible velocity field for a two-dimensional, incompressible flow, where U is a constant velocity, ω is a constant frequency, x and y are Cartesian co-ordinates, and t is the time.

Find the equation of the streamline through the point $\left(\dfrac{U}{\omega},\dfrac{U}{\omega}\right)$ at $t=0$, and show that the path of the particle which is at $\left(\dfrac{U}{\omega},\dfrac{U}{\omega}\right)$ at $t=0$ is $x=\dfrac{U}{\omega}\,e^{\frac{1}{2}\omega^2t^2}$, $y=\dfrac{U}{\omega}\left(3e^{-\frac{1}{2}\omega^2t^2}-2\right)+U\omega t^2$.

15. An incompressible fluid of viscosity μ flows steadily between two co-axial circular cylinders of radii a and $b(a < b)$, the flow being everywhere parallel to the axis of the cylinders. Given that the pressure gradient is P determine the magnitude $v(r)$ of the fluid velocity at radius $r(a < rb)$ and hence find the downstream forces acting per unit length on the inner and outer cylinders.

16. Let (x, y) be a two-dimensional continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} kx^\alpha + 3xy^2 & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and k is constant. Given that the first quartile of x is at $1/2$, determine α and k.

PART-III

Theoretical Physics

Syllabus

1. **Classical Mechanics**

   Mechanics of a particle and system of particles−conservation laws− scattering in a central field − Lagrange's equation and their applications. Hamilton's equation, canonical transformation − special theory of relativity, small oscillation, vibration & accoustics.

2. **Electromagnetic theory**

   Classical electrodynamics, Maxwell's equations − gauge transformation − Poynting's theorem − wave equation and plane waves − radiating system and scattering.

3. **Statistical Physics & Condensed Matter Physics**

   Thermodynamic equilibrium, partition functions, density matrix, phase transition, spin systems, Models. Statistical fluctuations, Band theory of electrons, Semiconductor Physics.

4. **Quantum Mechanics and Quantum Field Theory**

   Inadequacy of classical physics − Schrodinger wave equation − general formalism of wave mechanics − exactly soluble eigenvalue problems. Approximation methods − scattering theory − time dependent perturbation theory. Symmetries and Conservation Laws : Relativistic equations : Klein-Gordon/Dirac equations, Lagrangian field theory, Examples of quantum field theory − ϕ^4, Quantum electrodynamics.

5. **Elementary Particles**

   Elementary particles − weak and strong interactions − selection rules − CPT theorem − Symmetry Principles in Particle Physics.

Sample questions for Theoretical Physics

1. a) Consider a particle constrained to move on the surface of an inverted cone of half angle α. The particle is subject to a gravitational field. Take the apex of the cone to be the origin, with its symmetry axis along the z-axis. By choosing the cylindrical coordinates as the generalized coordinates,

   (i) set up the Lagrangian of the system,

   (ii) show that the angular momentum about the z-axis is constant in magnitude, and

(iii) derive the equation of motion for r coordinate.

b) A 1×10^6 kg rocket has 1×10^3 kg of fuel on board. The rocket is parked in space when it suddenly becomes necessary to accelerate. The rocket engines ignite, and the 1×10^3 kg fuel are consumed. The exhaust (burnt fuel) is ejected during a very short time interval at a speed of $c/2$ relative to the inertial reference frame in which the rocket is initially at rest where c is the velocity of light in vacuum. Calculate the change in the mass of the rocket-fuel system.

2. A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction $\vec{B}$, the vector $\vec{B}$ being perpendicular to the plane (x, y). At time $t = 0$, a connector starts sliding translation wise from the vertex of the parabola with a constant acceleration f. Find the e.m.f. of electromagnetic induction in the loop thus formed as a function of y.

3. a) Prove that the change in temperature by the perfect gas during an adiabatic expansion (pressure change $P_1 - P_2 = dP$) at temperature T is given by

$$dT = \frac{T}{C_P}\left(\frac{\partial V}{\partial T}\right)_P dP = \frac{\alpha T V}{C_P}\, dP$$

where α is the coefficient of volume expansion and the other symbols have their usual meanings.

b) Two levels in an atom whose nuclear spin is $I = 3$, have the designations $^2D_{3/2}$ and $^2P_{1/2}$. Find the expected number of components in the hyperfine structure of the corresponding spectral line.

4. If the potential energy function for the NaCl crystal structure is given by

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^n} \qquad \alpha,\ \beta,\ n \quad \text{being} \quad \text{constants}$$

a) find out the intermolecular separation $r = r_0$ for the equilibrium condition. Hence find out the minimum potential energy.

b) Prove the following relation

$$\frac{1}{K_0} = \frac{\alpha(n-1)}{18 N r_0^4}$$

where K_0 is the compressibility of NaCl solid at $0^0 K$ at the equilibrium separation and volume of the solid is $V = 2Nr^3$.

5. (a) A light beam is propagating through a block of glass with refractive index 1.5. If the block is moving with constant velocity in the same direction as the beam, what is the velocity of light in the block as measured by an observer in the laboratory?

(b) A particle of mass $m_1 = 1gm$ travelling at a speed $.9c$, c being the speed of light, collides head on with a stationary particle of mass $m_2 = 10gm$ and is embedded in it. What is the rest mass and velocity of the resulting composite particle?

6. (a) If the effective density of states in valence band is eight times that in convection band in a pure semiconductor at 27^oC, find the shift of Fermi level from the middle of the energy gap assuming low concentration of electrons and holes in the semicondctor. (Boltzmann's constant $K_B = 1.33 \times 10^{-23} J/$ mole^o K, In $2 = .693$)

(b) Consider a line of $2N$ ions of alternating charges $\pm q$ with a repulsive potential A/R^n between nearest neighbours in addition to the usual Coulomb potential. Neglecting surface effects find the equilibrium separation R_0 for such a system. Let the crystal be compressed so that R_0 becomes $R_0(1-\delta)$. Calculate work done in compressing a unit length of the crystal to order δ^2.

7. a) A hypothetical semi-conductor has a conduction band (cb) that can be described by $E_{cb} = E_1 - E_2 \cos(ka)$ and valence band (vb) which is represented by $E_{vb} = E_3 - E_4 \sin^2\left(\frac{ka}{2}\right)$ where $E_3 < (E_1 - E_2)$ and $-\pi/a < k < \pi/a$. Find out the expressions for

i) the band-widths of the conduction band and the valence band,

ii) the band-gap of the material,

iii) the effective mass of the electrons at the bottom of the conduction band.

(b) A beam of electrons with kinetic energy 1 keV is diffracted as it passes through a polycrystalline metal foil. The metal has a cubic crystal structure with a spacing of $1A^0$. Calculate the wave length of electron and the Bragg angle for the first order diffraction. Take m, h, c the mass of the electron, Planck's constant and speed of light respectively as follows: $mc^2 = .5$Mev, $c = 3 \times 10^8$m/s, $h = 6.6 \times 10^{-34}$Js. Also take $1eV = 1.6 \times 10^{-19}$J.

8. a) The wave-function of a hydrogen-like atom is

$$\psi(r,\theta) = Z^{3/2}(6 - Zr)Zr\ e^{-Zr/3}\cos\theta.$$

What are the values of the quantum numbers n, l and m for this state? Obtain the wave-function of state with the same values of n and l but m increased by one unit.

b) The parity operator is defined by $\mathbf{r} \to -\mathbf{r}$. How does the electron wave-function in a hydrogen atom transform under parity operation?

9. a) Consider a Lagrangian density,

$$L = \frac{1}{2}\, \partial_\mu \phi \partial^\mu \phi - \frac{\mu^2}{2}\, \phi^2 + \frac{1}{2\lambda}\, \sigma^2 - \frac{1}{2}\, \sigma \phi^2,$$

where $\sigma(x)$ and $\phi(x)$ are two scalar fields. Use equations of motion to eliminate $\sigma(x)$ and determine the equivalent Lagrangian, that depends only on $\phi(x)$.

b) A Lagrangian for a scalar field $\phi(x)$ is given by,

$$L = \frac{1}{2}\, \partial_\mu \phi \partial^\mu \phi - \frac{a}{b}\, (1 - cos(b\phi))\,,$$

where a and b are constants.

(i) Derive the equation of motion for ϕ.

(ii) Expand L in powers of ϕ and identify the mass and quartic coupling constant in terms of a and b.

(iii) Find the Hamiltonian (energy) density if ϕ-field is independent of time.

10. Consider the theory of Dirac fermions interacting with photons (QED).

i) Find expression for the charge current density and show that it is conserved.

ii) Using Wick's theorem, derive amplitudes for the following processes:

(a) electron - positron annihilation process of $O(e^2)$ where e is the coupling constant.

(b) Same process of $O(e^4)$.

(c) Draw the Feynman diagrams of the above processes.

iii) Consider the energy projection operators $\Lambda^{\pm}(p) = \pm \frac{\not{p} + mc}{2mc}$, where $\not{p} = \gamma_\mu p^\mu$. Prove the following identities :

(a) $\left(\Lambda^{+}\right)^2 = \Lambda^{+}$

(b) $\Lambda^{+}\Lambda^{-} = 0$

11. a) The following two decays are forbidden. State the reason;

i) $p \to \pi^0 + e^+$

ii) $n \to \pi^+ + \mu^-$

The following two decays are allowed. Give justifications;

i) $n \to p + e^- + \overline{\nu}_e$

ii) $\pi^+ \to \mu^+ + \nu_\mu$

b) Obtain the exact ratio of the energy released when 1 gm of Uranium undergoes fission to the energy released when 1 gm of TNT explodes.

12. a) The Hamiltonian of a system is $\hat{H} = \epsilon \vec{\sigma}.\vec{n}$, where ϵ is a constant having the dimension of energy, $\vec{n}$ is an arbitrary unit vector and σ_x, σ_y and σ_z are the Pauli matrices.

Find the energy eigenvalues and normalized eigenvectors of $\hat{H}$.

b) If (q, p) are generalized coordinates then show that the following transformation is canonical,

$$Q = \log\left(\frac{1}{q}\sin p\right), \; P = q \cot p$$

PART-IV

Statistics
Syllabus

Probability and Sampling Distributions : Notion of sample space, combinatorial probability, conditional probability and independence, random variable and expectation, moments, standard discrete and continuous distributions, sampling distribution of statistics based on normal samples.

Descriptive Statistics (including Numerical Analysis) : Descriptive measures, graduation of frequency curves, correlation and regression (bivariate and multivariate), polynomial interpolation, numerical integration.

Inference : Elementary theory and methods of estimation (unbiasedness, minimum variance, sufficiency, mle, method of moments). Testing of hypotheses (basic concepts and simple applications of Neyman-Pearson lemma).

Designs (including elementary ANOVA) and Sample Surveys : Basic designs, (CRD/RBD/LSD) and their analysis, conventional sampling techniques (srswr/wor) including stratification.

Sample Questions

1. The standard deviation of two sets containing n_1 and n_2 members are σ_1 and σ_2 respectively, being measured from their respective means m_1 and m_2. If the two sets are grouped together as one set of $(n_1 + n_2)$ members, show that the standard deviation σ of this set measured from its mean is given by

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(m_1 - m_2)^2.$$

2. (a) A pair of dice is thrown. If it is known that one dice shows a 4, calcutate the probability that the total of both the dice is greater than 7.

   (b) Let X have a probability distribution with a density at x as

$$\begin{aligned} f(x) &= k_0\sqrt{x}, \qquad 0 < x < 1 \\ &= 0, \quad \text{elsewhere} \end{aligned}$$

   Calcutate the probability $P(0.3 < X < 0.6)$, numerically.

3. A ball is drawn from an urn containing 9 balls numbered 0, 1, 2, $\cdots$, 8, of which the first 4 are white, the next 3 red and the last 2 black. If the colours white, red and black are reckoned as colour numbered 0, 1 and 2 respectively, find the joint distribution of the random variables $-$ the number on the ball drawn and its colour number.

4. Find the maximum likelihood estimator of θ for a random sample from a distribution having the p.d.f.

$$f_\theta(x) = \begin{cases} (\theta+1)x^\theta & \text{if } \; 0 \leq x \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find also the estimator of θ by the method of moments.

5. Describe the model and analysis used in Randomised Block Design. Explain the utility of Latin Square Design. Construct a 4×4 Latin Square Design.

6. Let X_1 X_2 be two iid Bernoulli random variables with probability of success p. Find the most powerful test for testing $H_0 : p = \dfrac{1}{2}$ against $H_1 : p = \dfrac{2}{3}$ at level of significance $\alpha = .05$.

7. Let X, Y have the joint p.d.f.

$$f(x,y) = \begin{cases} \dfrac{6-x-y}{8} & \text{if } \; 0 \leq x \leq 2, \; 2 \leq y \leq 4; \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X+Y<3)$ and $P(X<1|Y=3)$.

8. Consider a multiple choice test of 20 questions, each with 5 choices. A candidate scores 1 for correct answer and 0 otherwise. What would be the most likely score for a candidate who guesses each time ? What would be the mean score for such students ? If a particular candidate is able to answer 70% of the questions correctly and guesses the rest, find the mean score of such candidates.

9. To estimate the total number of workers employed in an industry comprising 90 factories in all, a sample of 10 factories are selected in the following manner : The smallest two factories are included in the sample and in addition a simple random sample (without replacement) of size 8 is selected from the remaining 88 factories. The estimator proposed is $90\bar{y}$ where $\bar{y}$ is the sample mean of number of workers of all the 10 factories in the sample.

   Show that the estimator is not unbiased. Suggest an unbiased estimator and prove that your estimator is unbiased.

10. Let X_1 and X_2 constitute a random sample of size 2 from the population with a density of the form $f(x|\theta) = \theta x^{\theta-1}$ for $0 < x < 1$.

    If the critical region "$x_1 x_2 \geq \frac{3}{4}$" is used to test the null hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$, what is the power of this test at $\theta = 2$?

11. Let the probability p_n that a family has exactly n children be αp^n where $n \geq 1$ and $p_0 = 1 - \alpha p(1+p+p^2 \cdots)$. Suppose that all sex distributions of n children have the same probability. For a positive integer k, find the probability that a family has exactly k boys.

12. In estimating the mean of a finite survey population on drawing a sample of a given size from it explain how one may involve an 'analysis of variance' arrgument to justify the efficicacy of stratified sampling. Develop a principle in formation of strata. If you are given a stratified simple random sample taken without replacement independently stratum-wise, show how you may derive an unbiased estimator for the population variance.

13. From a population of N units of varying sizes $X(i = 1, 2, \cdots, N)$, a sample of size n is selected with probability of selection of a unit proportional to the size and with replacement. A technician calculates the sample mean y of the observed y value under study. Is y unbiased for the population mean of the y values ? If so, verify the unbiased property. If not, calculate the bias.

14. Based on a simple random sample of size 200, a 95% confidence interval for the population mean turned out to be (10, 20). Find a 90% confidence interval for the population mean based on this information.

15. Let X_i, $i = 1, 2, 3, 4$ be iid $N(0, 1)$ random variables. Find the density function of $Z = X_1X_2 - X_3X_4$.