Test Code : RP : (Short Answer type) 2010

Junior Research Fellowship in Theoretical Physics and Applied Mathematics

The candidates for Junior Research Fellowships in Applied Mathematics and Theoretical Physics will have to take two tests−Test MIII (objective type) in the forenoon session and Test RP (short answer type) in the after noon session.

The RP test booklet will consist of two parts. The candidates are required to answer from Part I and only one of the remaining parts II, III.

The syllabus and sample questions for the test are as follows.

PART-I

Mathematical and logical reasoning

Syllabus

B.Sc. Pass Mathematics syllabus of Indian Universities.

Sample Questions

1. A function $f(x)$ satisfies the following relation

$$af(x) + 2f\left(\frac{1}{x}\right) = x - 2,\ x \neq 0,\ a \neq 2.$$

Show that $f(2) = \dfrac{3}{a^2 - 4}$.

2. Let f be a real valued function defined on the interval $[-2, 2]$ as:

$$f(x) \ = \ \begin{cases} (x+1)2^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

i) Find the range of the function.

ii) Is f continuous at every point in $(-2, 2)$? Justify your answer.

3. A point moves in the (x, y)-plane with position $(x(t), y(t))$ at time t, where $x(t) = 1/t$ and $y(t) = \dfrac{1}{\sqrt{2}}(1 + t)$ for $t > 0$. Show that the point comes closest to the origin when $t = 1$.

4. The position of a particle moving in a plane is given by $x = \sin \omega t$, $y = \cos \alpha\omega t$. Show that the trajectory repeats itself periodically only if α is a rational number.

5. Evaluate $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

6. X is a uniformly distributed random variable with probability density function

$$f(x) = \begin{cases} \dfrac{5}{a} & \text{for} \quad -\frac{a}{10} \le x \le \frac{a}{10} \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

where a is a nonnegative constant. If $P(|x| < 2) = 2P(|x| > 2)$, then find a.

7. Find the roots of the equation $z^5 = -i$, and indicate their locations in the complex plane.

8. The rate of cooling of a substance in moving air is proportional to the difference of temperatures of the substance and the air. A substance cools from 36^0C to 34^0C in 15 minutes. Find the time required to cool the substance from 36^0C to 32^0C, considering the constant temperature of air as 30^0C.

9. Assuming nothing else blocks their view, how far can two people, each of hight h (in meters), walk from each other until they can no longer see each other due to earth's curvature. (Take earth to be a perfect sphere with radius R (in meters)).

10. Evaluate $\int_0^{\pi/2} \frac{dx}{1 + (tanx)^{100}}$.

11. Let $f(x, y) = |xy|^{\frac{1}{2}}$. Is this function differentiable at (0,0)?

12. If $f(x, f(y)) = x^p y^q$ all x, y, then show that $p^2 = q$ and find $f(x)$.

13. Show that the area of the triangle formed by z, iz, and $z + iz$ is $\dfrac{r^2}{2}$ where $r = |z|$ and $z = a + ib$, a, b being real nonzero numbers.

14. A body of mass m thrown at an angle α to the horizontal plane with an initial velocity v_0, moves under the action of the force of gravity and the force of resistance $\vec{R}$ of the air given by $\vec{R} = \gamma \vec{v}$ where $\vec{v}$ is the velocity vector and γ is a constant. Determine the maximum height attained by the body.

15. A particle of mass m moves under a conservative force with potential energy $V(x) = \dfrac{Cx}{(a^2 + x^2)}$, where C and a are positive constants. Find the position of stable equilibrium and the equation of motion if there is a small oscillation about the stable position.

16. Find the maximum possible value of xy^2z^3 subject to the conditions x, y, $z \geq 0$ and $x + y + z = 3$.

17. A point moves in the (x, y)-plane such that at time $t (> 0)$ it has the coordinates $(1/t, (1 + t)/\sqrt{2})$. Find when it comes closest to the origin.

PART-II

Applied Mathematics

Syllabus

1. *Abstract algebra* : Groups, rings, fields.

2. *Real analysis* : Functions of single and several variables, metric space, normed linear space, Riemann Integral, Fourier series.

3. *Differential equations* : ODE − Existence of solution, fundamental system of integrals, elementary notions, special functions. PDE upto second order, equations of parabolic, hyperbolic and elliptic type.

4. *Dynamics of particles and rigid bodies* : Motion of a particle in a plane and on a smooth curve under different laws of resistance, kinematics of a rigid body, motion of a solid body on an inclined smooth or rough plane.

5. *Functions of complex variables* : Analytic function, Cauchy's theorem Taylor and Laurent series, singularities, branch-point, contour integration, analytic continuation.

6. *Fluid Mechanics* : Kinematics of fluid, equation of continuity, irrotational motion, velocity potential, dynamics of ideal fluid, Eulerian and Lagrangian equations of motion, stream function, sources, sinks and doublets, vortex, surface waves, group velocity, Viscous flow − Navier Stokes equation, boundary layer theory, simple problems.

7. *Probability and statistics* : Probability axioms, conditional probability, probability distribution, mathematical expectations, characteristic functions, covariance, correlation coefficient. Law of large numbers, central limit theorem. Random samples, sample characteristics, estimation, statistical hypothesis, Neyman pearson theorem, likelihood ratio testing.

Sample Questions

1. Let m and n be positive integers and F is a field. Let f_i $(i = 1, 2, \ldots, m)$ be linear functionals on F^n. For α in F^n, let $T\alpha = (f_1(\alpha), f_2(\alpha), \ldots\ \ldots, f_m$ $(\alpha))$. Show that T is a linear transformation from F^n into F^m. Also show that every linear transformation from F^n into F^m is of the above form for some f_i $(i = 1, 2, \ldots, m)$.

2. (a) Let $X_1 = [1, 2]$ and $X_2 = [0, 1]$. Let d_1 denote the Euclidean metric in X_1 and let $d_2(x, y) = 2|x - y|$ in X_2. Show that (X_1, d_1) and (X_2, d_2) are equivalent metric spaces.

b) Two different metrics on the space $X = x \in R : 0 < x \leq 1$ are defined by $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\frac{1}{x} - \frac{1}{y}|$. Are the spaces (X, d_1) and (X, d_2) equivalent? Give reasons for your answer.

3. (a) A particle of mass m moves on the outside of a smooth circle of radius a kept vertically. The particle starts from a point P at an angular distance α from the highest point A and is allowed to slide down. Show that the particle leaves the circle at point Q whose angular distance θ from A is given by $\cos\theta = \frac{2}{3}\cos\alpha$.

   (b) An ice cube is melting, becoming smaller but still cubical.The rate of melting is proportional to the surface area of the cube. Let $v(t)$ be the volume of the cube at time t. Explain why $\frac{dv}{dt} = \alpha v^{\frac{2}{3}}$ for some constant α. Will α be positive or negative.

4. Show that if the solution of the ODE

$$2xy'' + (3 - 2x)y' + 2y = 0$$

is expressed in the form $y = \sum_{n=0}^{\infty} a_n\, x^{n+\sigma}$, σ can take two possible values. Find the relation between a_n and a_{n+1}, and show that one solution reduces to a polynomial.

5. Show that the function defined by

$$\begin{aligned} f(x, y) &= \frac{x^5 + y^5}{x^2 + y^2} \quad \text{for} \quad (x, y) \neq (0, 0) \\ &= 0 \quad \text{for} \quad (x, y) = (0, 0) \end{aligned}$$

is differentiable at all points. Find every stationary point of this function, and determine whether each is a maximum, minimum or a saddle point.

6. The function $u(x, t)$ satisfies the differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \quad t > 0$$

subject to the boundary conditions $u(x, t) \to 0$ as $|x| \to \infty$ and the initial conditions

$$\begin{aligned} u(x, 0) &= -e^{x} \quad & x < 0 \\ &= 0 \quad & x = 0 \\ &= e^{-x} \quad & x > 0 \end{aligned}$$

. Show by Fourier transform that for $t > 0$,

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{ke^{-k^2t} sinkx}{k^2+1}dk$$

.

7. a) Find the Fourier series of a function $f(x)$ defined as follows

$$f(x) = \left\{ \begin{array}{ll} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{array} \right\}$$

b) If $f(z)$ $(z = x + iy)$ is an analytic function, prove that $\log|f(z)|$ is harmonic.

8. Let $f(x)$ be defined on $[1, \infty)$. Also $f(x)$ is continuous and differentiable on that interval and further the derivative $f'(x)$ is given by $f'(x) = \dfrac{1}{x^2 + (f(x))^2}$. Show that for all $x \geq 1$, $f(x) \leq 1 + \dfrac{\pi}{4}$.

9. Examine the extremum values from

$$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x = 0.$$

10. Calculate the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Show that the eigenvectors are orthogonal and form a basis in C^2. Find also a unitary matrix U.

11. Find the integral surface of the equation

$$(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$$

passing through the curve $xz = a^3$, $y = 0$, where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

12. Evaluate the integral $\displaystyle\int_c \frac{e^{1/z^2}}{z^2+1}dz$, where c is the curve given by $|z - i| = \dfrac{7}{2}$, the integration being taken counterclockwise.

13. Steam is rushing from a boiler through a conical pipe, the diameters of the ends being D and d. If V and v be the corresponding velocities of the stream and if the motion be supposed to be that of divergence from the vertex of the cone, prove that

$$\frac{v}{V} = \frac{D^2}{d^2}e^{(v^2-V^2)/3k}$$

where k is pressure divided by density and taken as constant.

14. An incompressible fluid of viscosity μ flows steadily between two co-axial circular cylinders of radii a and $b(a < b)$, the flow being everywhere parallel to the axis of the cylinders. Given that the pressure gradient is P determine the magnitude $v(r)$ of the fluid velocity at radius $r(a < rb)$ and hence find the downstream forces acting per unit length on the inner and outer cylinders.

15. The continuous random variable X has the probability density function

$$f(x) = \begin{cases} axe^x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of the constant a. Given that $Y = e^{X-1}$, derive the probability density function of Y and find $E(Y)$.

PART-III

Theoretical Physics

Syllabus

1. **Classical Mechanics**

   Mechanics of a particle and system of particles−conservation laws− scattering in a central field − Lagrange's equation and their applications. Hamilton's equation, canonical transformation − special theory of relativity, small oscillation, vibration & accoustics.

2. **Electromagnetic theory**

   Electrostatics, Magnetostatics, Classical electrodynamics, Maxwell's equations − gauge transformation − Poynting's theorem − wave equation and plane waves − radiating system and scattering.

3. **Statistical Physics & Condensed Matter Physics**

   Thermodynamic equilibrium, partition functions, density matrix, phase transition, spin systems, Models. Statistical fluctuations, Band theory of electrons, Semiconductor Physics.

4. **Quantum Mechanics and Quantum Field Theory**

   Inadequacy of classical physics − Schroedinger wave equation − general formalism of wave mechanics − exactly soluble eigenvalue problems. Approximation methods − scattering theory − time dependent perturbation theory. Symmetries and Conservation Laws : Relativistic equations : Klein-Gordon/Dirac equations, Lagrangian field theory, Examples of quantum field theory − ϕ^4, Quantum electrodynamics.

5. **Elementary Particles**

   Elementary particles − weak and strong interactions − selection rules − CPT theorem − Symmetry Principles in Particle Physics.

Sample questions for Theoretical Physics

1. a) Consider a particle constrained to move on the surface of an inverted cone of half angle α. The particle is subject to a gravitational field. Take the apex of the cone to be the origin, with its symmetry axis along the z-axis. By choosing the cylindrical coordinates as the generalized coordinates,

   (i) set up the Lagrangian of the system,

   (ii) show that the angular momentum about the z-axis is constant in magnitude, and

(iii) derive the equation of motion for r coordinate.

b) A 1×10^6 kg rocket has 1×10^3 kg of fuel on board. The rocket is parked in space when it suddenly becomes necessary to accelerate. The rocket engines ignite, and the 1×10^3 kg fuel are consumed. The exhaust (burnt fuel) is ejected during a very short time interval at a speed of $c/2$ relative to the inertial reference frame in which the rocket is initially at rest where c is the velocity of light in vacuum. Calculate the change in the mass of the rocket-fuel system.

2. A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction $\vec{B}$, the vector $\vec{B}$ being perpendicular to the plane (x, y). At time $t = 0$, a connector starts sliding translation wise from the vertex of the parabola with a constant acceleration f. Find the e.m.f. of electromagnetic induction in the loop thus formed as a function of y.

3. a) Prove that the change in temperature by the perfect gas during an adiabatic expansion (pressure change $P_1 - P_2 = dP$) at temperature T is given by

$$dT = \frac{T}{C_P}\left(\frac{\partial V}{\partial T}\right)_P dP = \frac{\alpha T V}{C_P}\, dP$$

where α is the coefficient of volume expansion and the other symbols have their usual meanings.

b) Two levels in an atom whose nuclear spin is $I = 3$, have the designations ${}^2D_{3/2}$ and ${}^2P_{1/2}$. Find the expected number of components in the hyperfine structure of the corresponding spectral line.

4. If the potential energy function for the NaCl crystal structure is given by

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^n} \qquad \alpha,\ \beta,\ n \quad \text{being} \quad \text{constants}$$

a) find out the intermolecular separation $r = r_0$ for the equilibrium condition. Hence find out the minimum potential energy.

b) Prove the following relation

$$\frac{1}{K_0} = \frac{\alpha(n-1)}{18 N r_0^4}$$

where K_0 is the compressibility of NaCl solid at $0^0 K$ at the equilibrium separation and volume of the solid is $V = 2Nr^3$.

5. (a) It is known from special theory of relativity that the Doppler shift is given $\lambda' = \lambda\sqrt{(1 + V/c)\,(1 - V/c)}$ where λ is the emitted wavelength as seen in a reference frame at rest with respect to the source, and λ' is the wavelength measured in a frame moving with velocity V away from the source along the line of sight. Show that the Doppler shift in wavelength is $z \approx V/c$ for $V < c$ with c being the velocity of light.

(b) A spaceship is moving away from the earth with speed $v = 0.8c$. When the ship is at a distance of 6.66×10^8km from earth as measured in the earth's reference frame, a radio signal is sent out to the spaceship by an observer on earth. How long will it take for the signal to reach the ship :

   a) As measured in the ship's reference frame ?
   b) As measured in the earth's reference frame ?
   c) Also give the location of the spaceship when the signal is received in both frames. In the above questions $c =$ velocity of light.

6. (a) If the effective density of states in valence band is eight times that in convection band in a pure semiconductor at 27^oC, find the shift of Fermi level from the middle of the energy gap assuming low concentration of electrons and holes in the semicondctor. (Boltzmann's constant $K_B = 1.33 \times 10^{-23} J/$ mole^o K, In $2 = .693$)

   (b) Consider a line of $2N$ ions of alternating charges $\pm q$ with a repulsive potential A/R^n between nearest neighbours in addition to the usual Coulomb potential. Neglecting surface effects find the equilibrium separation R_0 for such a system. Let the crystal be compressed so that R_0 becomes $R_0(1-\delta)$. Calculate work done in compressing a unit length of the crystal to order δ^2.

7. a) A cylindrical capacitor has an inner conductor of radius r_1 and an outer conductor of radius r_2. The outer conductor is grounded and the inner conductor is charged so as to have a potential V_0. In terms of V_0, r_1 and r_2, find

   (i) electric field at r for $r_1 < r < r_2$.
   (ii) the potential at some r.

   b) A beam of electrons with kinetic energy 1 keV is diffracted as it passes through a polycrystalline metal foil. The metal has a cubic crystal structure with a spacing of $1A^0$. Calculate the wave length of electron and the Bragg angle for the first order diffraction. Take m, h, c the mass of the electron, Planck's constant and speed of light respectively as follows: $mc^2 = .5$Mev, $c = 3 \times 10^8$m/s, $h = 6.6 \times 10^{-34}$Js. Also take $1eV = 1.6 \times 10^{-19}$J.

8. a) Show that for the Dirac Hamiltonian $H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2 + v(r)$, the total angular momentum is a constant of motion i.e., $\frac{d\vec{J}}{dt} = 0$.

   b) Parity operator is defined as $p\psi(x, y, z) = \psi(-x, -y, -z)$. What are corresponding definition for $\psi(r, \theta, \phi)$ and $\psi(r, \theta, z)$? Show that the following states are parity eigenstates and determine their parity.

$$\psi_1 = A(x+y+z)\; e^{-(x^2+y^2+z^2)}$$

$$\psi_2 = Br\; e^{-r^2} \cos\theta$$

$$\psi_3 = c\frac{\sqrt{\rho^2+z^2}}{z^5} \sin\theta$$

9. a) Consider the Lagrangian of a scalar field theory with the self interaction term $\lambda\phi^5$.
i) Find out the equation of motion and the Hamiltonian.
ii) Consider a process where three ϕ-particles collide and two ϕ-particles come out after the collision. Draw possible Feynman diagrams for the above process having amplitude proportional to λ and λ^3.
iii) If one tries to quantize the theory in $3+1$ (physical) spacetime what are the problems one will face?

b) Consider the Dirac equation

$$(i\hbar\gamma^\mu \frac{\partial}{\partial x_\mu} - mc)\psi(x) = 0.$$

The Projection operators are defined as,

$$\Lambda^+(\vec{p}) = \frac{+\gamma^\mu p_\mu + mc}{2mc} \quad ; \quad \Lambda^-(\vec{p}) = \frac{-\gamma^\mu p_\mu + mc}{2mc}.$$

Prove the identities,

$$\Lambda^+(\vec{p})^2 = \Lambda^+(\vec{p}) \quad ; \quad \Lambda^+(\vec{p})\Lambda^-(\vec{p}) = 0.$$

10. Consider the real free Klein-Gordon Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - m^2\phi^2\right).$$

Using the expansion,

$$\phi(x) = \sum_k \sqrt{\frac{\hbar c^2}{2V\omega_k}} \left(a(\vec{k})e^{-ik^\mu x_\mu} + a^+(\vec{k})e^{ik^\mu x_\mu}\right)$$

show that the Hamiltonian is given by

$$H = \sum_k \hbar\omega_k \left[a^+(\vec{k})a^+(\vec{k}) + \frac{1}{2}\right].$$

Note: $k^\mu x_\mu = \omega_k t - \vec{k}.\vec{x}$.

11. Consider the theory of Dirac fermions interacting with photons (QED).

  i) Find expression for the charge current density and show that it is conserved.

  ii) Using Wick's theorem, derive amplitudes for the following processes:

    (a) electron - positron annihilation process of $O(e^2)$ where e is the coupling constant.

    (b) Same process of $O(e^4)$.

    (c) Draw the Feynman diagrams of the above processes.

  iii) Consider the energy projection operators $\Lambda^{\pm}(p) = \pm\frac{\not{p} + mc}{2mc}$, where $\not{p} = \gamma_\mu p^\mu$. Prove the following identities :

    (a) $(\Lambda^+)^2 = \Lambda^+$

    (b) $\Lambda^+\Lambda^- = 0$

12. a) State with reason which of the following reactions are admissible:

    i) $\mu^+ \to e^+\overline{\nu}_\mu$,

    ii) $\pi^- p \to K^0 n$,

    iii) $\pi^+ p \to n\pi^0$,

    iv) $p \to ne^+\nu_e$,

    v) $\pi^+ p \to pe^+$,

    vi) $p \to ne^+\nu_e$,

    vii) $\pi^- p \to K^0\Lambda^0$.

  b) Obtain the exact ratio of the energy released when 1 gm of Uranium undergoes fission to the energy released when 1 gm of TNT explodes.

13. a) The Hamiltonian of a system is $\hat{H} = \epsilon\vec{\sigma}.\vec{n}$, where ϵ is a constant having the dimension of energy, $\vec{n}$ is an arbitrary unit vector and σ_x, σ_y and σ_z are the Pauli matrices.

  Find the energy eigenvalues and normalized eigenvectors of $\hat{H}$.

  b) Show that the transformation

$$\begin{aligned} q &= \sqrt{2P}\sin\theta \\ p &= \sqrt{2P}\cos\theta \end{aligned}$$

  is a canonical transformation.

14. a) Two identical harmonic oscillators with spring constant k interact via a potential $\epsilon x_1 x_2$. Calculate the exact energy eigenvalues.

  b) Let $S_\pm = S_x \pm iS_y$ where S_x, S_y and S_z are Pauli spin matrices. Then find $S_\pm|\pm, \frac{1}{2} >$ where $|\pm, \frac{1}{2} >$ are eigenvectors of S_z.