Test Code : RP : (Short Answer type) 2007

Junior Research Fellowship in Theoretical Physics and Applied Mathematics

The candidates for Junior Research Fellowships in Applied Mathematics and Theoretical Physics will have to take two tests−Test MIII (objective type) in the forenoon session and Test RP (short answer type) in the after noon session.

The RP test booklet will consist of two parts. The candidates are required to answer from Part I and only one of the remaining parts II, III and IV.

The syllabus and sample questions for the test are as follows.

PART-I

Mathematical and logical reasoning

Syllabus

B.Sc. Pass Mathematics syllabus of Indian Universities.

Sample Questions

1. Let f be a real valued function defined on the interval $[-2, 2]$ as:

$$f(x) \;=\; \begin{cases} (x+1)2^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & \text{for} \quad x \neq 0 \\ 0 & \text{for} \quad x = 0 \end{cases}$$

   i) Find the range of the function.

   ii) Is f continuous at every point in $(-2, 2)$? Justify your answer.

2. Let $g : R \to R$ be a continuous function such that $g(x) = g\left(\frac{x-1}{2}\right)$ for all x. Show that g must be a constant function.

3. Find the minimum value of

$$2^{\cos x} + 2^{\sin x} \qquad (0 \leq x \leq 2\pi)\,.$$

4. Evaluate $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

5. If $z_0,\ z_1,\ \cdots\ z_{n-1}$ are the roots of the equation $z^n - 1 = 0$, and $z_0 = 1$, then find the value of

$$(1 - z_1)\,(1 - z_2)\cdots(1 - z_{n-1})$$

6. X is a uniformly distributed random variable with probability density function

$$f(x) = \begin{cases} \frac{5}{a} & \text{for} \quad -\frac{a}{10} \leq x \leq \frac{a}{10} \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

where a is a nonnegative constant. If $P(|x| < 2) = 2P(|x| > 2)$, then find a.

7. Suppose a, b, c are positive integers such that

$$abc + ab + bc + ca + a + b + c = 1000$$

Find the value of $a + b + c$.

8. If $f(x)$ is continuous on $(0, \infty)$ and for $x \neq 0$

$$f(x) + 2f\left(\frac{1}{x}\right) = \frac{1}{x} - 2$$

find $\int_2^3 f(x)dx$.

9. If $f(x) \geq 0$ for $a \leq x \leq b$, we know that $\int\limits_a^b f(x)dx \geq 0$. However, by actual calculation we find that $\int\limits_{-2}^{1} \frac{dx}{x^2} = -\frac{3}{2}$, which is clearly absurd. What went wrong ?

10. Let $f(x, y) = |xy|^{\frac{1}{2}}$. Is this function differentiable at (0,0)?

11. Find the greatest and least values of the function

$$f(x) = x^3 - 3x^2 + 2x + 1$$

in the interval $[2, 3]$.

12. A point moves in the (x, y)-plane such that at time $t(> 0)$ it has the co-ordinates $(1/t, (1 + t)/\sqrt{2})$. Find when it comes closest to the origin.

13. A body of mass m thrown at an angle α to the horizontal plane with an initial velocity v_0, moves under the action of the force of gravity and the force of resistance $\vec{R}$ of the air given by $\vec{R} = \gamma\vec{v}$ where $\vec{v}$ is the velocity vector and γ is a constant. Determine the maximum height attained by the body.

14. If $f(x, f(y)) = x^p y^q$ all x, y, then show that $p^2 = q$ and find $f(x)$.

15. Show that the area of the triangle formed by z, iz and $z + iz$ is $\frac{r^2}{2}$ where $r = |z|$ and $z = a + ib$, a, b being real nonzero numbers.

16. Find the maximum possible value of xy^2z^3 subject to the conditions x, y, $z \geq 0$ and $x + y + z = 3$.

17. If the lines $3x - 4y + 4 = 0$, $6x - 8y - 7 = 0$ are tangents to the same circle, find the radius of the circle.

18. A particle of mass m moves under a conservative force with potential energy $V(x) = \dfrac{Cx}{(a^2 + x^2)}$, where C and a are positive constants. Find the position of stable equilibrium and the equation of motion if there is a small oscillation about the stable position.

PART-II

Applied Mathematics

Syllabus

1. *Abstract algebra* : Groups, rings, fields.

2. *Real analysis* : Functions of single and several variables, metric space, normed linear space, Riemann Integral, Fourier series.

3. *Differential equations* : ODE − Existence of solution, fundamental system of integrals, elementary notions, special functions. PDE upto second order, equations of parabolic, hyperbolic and elliptic type.

4. *Dynamics of particles and rigid bodies* : Motion of a particle in a plane and on a smooth curve under different laws of resistance, kinematics of a rigid body, motion of a solid body on an inclined smooth or rough plane.

5. *Functions of complex variables* : Analytic function, Cauchy's theorem Taylor and Laurent series, singularities, branch-point, contour integration, analytic continuation.

6. *Numerical analysis* : Solution of a system of linear equations, polynomial interpolation, numerical integration formula (Newton-Cotes type).

7. *Fluid Mechanics* : Kinematics of fluid, equation of continuity, irrotational motion, velocity potential, dynamics of ideal fluid, Eulerian and Lagrangian equations of motion, stream function, sources, sinks and doublets, vortex, surface waves, group velocity, Viscous flow − Navier Stokes equation, boundary layer theory, simple problems.

8. *Probability and statistics* : Probability axioms, conditional probability, probability distribution, mathematical expectations, characteristic functions, covariance, correlation coefficient. Law of large numbers, central limit theorem. Random samples, sample characteristics, estimation, statistical hypothesis, Neyman pearson theorem, likelihood ratio testing.

Sample Questions

1. (a) Let G be a group such that

$$(ab)^m = a^m b^m$$

for three consecutive integers m, $m+1$ and $m+2$ for all $a, b \in G$. Show that G is abelian.

(b) Let R be a ring with a unit element. Form another ring R' by defining

$$a \oplus b = a + b + 1, \quad a.b = ab + a + b$$

Determine the zero element and unit element of R'.

2. (a) Let $X_1 = [1, 2]$ and $X_2 = [0, 1]$. Let d_1 denote the Euclidean metric in X_1 and let $d_2(x, y) = 2|x - y|$ in X_2. Show that (X_1, d_1) and (X_2, d_2) are equivalent metric spaces.

   (b) Two different metrics on the space $X = x \in R : 0 < x \leq 1$ are defined by $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\frac{1}{x} - \frac{1}{y}|$. Are the spaces (X, d_1) and (X, d_2) equivalent? Give reasons for your answer.

3. (a) A particle of mass m moves on the outside of a smooth circle of radius a kept vertically. The particle starts from a point P at an angular distance α from the highest point A and is allowed to slide down. Show that the particle leaves the circle at point Q whose angular distance θ from A is given by $\cos\theta = \frac{2}{3}\cos\alpha$.

   (b) An ice cube is melting, becoming smaller but still cubical. The rate of melting is proportional to the surface area of the cube. Let $v(t)$ be the volume of the cube at time t. Explain why $\frac{dv}{dt} = \alpha v^{\frac{2}{3}}$ for some constant α. Will α be positive or negative.

4. Show that if the solution of the ODE

$$2xy'' + (3 - 2x)y' + 2y = 0$$

is expressed in the form $y = \sum_{n=0}^{\infty} a_n \, x^{n+\sigma}$, σ can take two possible values. Find the relation between a_n and a_{n+1}, and show that one solution reduces to a polynomial.

5. Solve the partial differential equation

$$u_{xx} + 7u_{xy} - 6u_{yy} = 5\sin(x + 2y) + 7e^{3x+y}$$

6. Reduce the partial differential equation

$$(a-1)^2\frac{\partial^2 u}{\partial x^2} - y^{2a}\frac{\partial^2 u}{\partial y^2} = ay^{a-1}\frac{\partial u}{\partial y}$$

to canonical form and find its general solution.

7. a) Show that

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) \quad \text{for} \quad 0 \leq x \leq \pi.$$

Deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.$$

b) If $f(z)$ $(z = x + iy)$ is an analytic function, prove that $\log |f(z)|$ is harmonic.

8. Find the Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$$

The period is 10. How should $f(x)$ be defined at $x = 5$ so that the Fourier series will converge to $f(x)$ for $-5 \leq x \leq 5$

9. By considering the contour integral $\int_C \left(\frac{z-1}{z+1}\right)^{\frac{1}{6}} dz$ where C is a simple closed contour that encircles the real interval $[-1, 1]$, show that

$$\int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{\frac{1}{6}} dx = \frac{2\pi}{3}$$

10. The function $u(t)$ satisfies the differential equation

$$t\frac{d^2u}{dt^2} + (2t+1)\frac{du}{dt} + u = 0$$

and $u(0) = 1$. Show that the Laplace transform of $u(t)$ is $(s^2 + 2s)^{-1/2}$. Explain why the condition $u(0) = 1$ is sufficient to determine a particular solution of this second-order differential equation.

11. Construct an integration rule of the form

$$\int_{-1}^{1} f(x)dx \simeq c_0 f\left(-\frac{1}{2}\right) + c_1 f(0) + c_2 f(\frac{1}{2})$$

which is exact for all polynomials of degree ≤ 2. Find the error term.

12. An incompressible fluid of viscosity μ flows steadily between two co-axial circular cylinders of radii a and b $(a < b)$, the flow being everywhere parallel to the axis of the cylinders. Given that the pressure gradient is P determine the magnitude $v(r)$ of the fluid velocity at radius r $(a < rb)$ and hence find the downstream forces acting per unit length on the inner and outer cylinders.

13. The area of cross section of a large tank is $0.5m^2$. It has an opening near the bottom having area of cross section $1cm^2$. A load of $20kg$ is applied on the water at the top. Find the velocity of water coming out of the opening at the time when the height of water level is $50cm$ above the bottom. Take $g = 10m/\text{sec}^2$.

14. A viscous fluid flows along a circular pipe with diameter D and length L. Assuming one dimensional flow, show that the pressure drop is given by

$$\Delta p = \frac{32\mu L\bar{u}}{D^2}$$

where $\bar{u}$ is the mean velocity of flow and μ is the viscocity of the fluid.

15. The random variables $X_1,\ X_2\ \ldots$ have $E(X_i) = 0$ for $i = 1, 2, \ldots$ and $E(X_iX_j) = \beta^{j-i}$ for $1 \leq i \leq j$ where $0 < \beta < 1$. If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, show that

$$\text{Var}\ \ (Y_n) = \frac{1+\beta}{n(1-\beta)} - \frac{2\beta(1-\beta^n)}{n^2(1-\beta)^2}$$

PART-III

Theoretical Physics

Syllabus

1. **Classical Mechanics**

   Mechanics of a particle and system of particles−conservation laws− scattering in a central field − Lagrange's equation and their applications. Hamilton's equation, canonical transformation − special theory of relativity, small oscillation, vibration & accoustics.

2. **Electromagnetic theory**

   Classical electrodynamics, Maxwell's equations − gauge transformation − Poynting's theorem − wave equation and plane waves − radiating system and scattering.

3. **Statistical Physics & Condensed Matter Physics**

   Thermodynamic equilibrium, partition functions, density matrix, phase transition, spin systems, Models. Statistical fluctuations, Band theory of electrons, Semiconductor Physics.

4. **Quantum Mechanics and Quantum Field Theory**

   Inadequacy of classical physics − Schrodinger wave equation − general formalism of wave mechanics − exactly soluble eigenvalue problems. Approximation methods − scattering theory − time dependent perturbation theory. Symmetries and Conservation Laws : Relativistic equations : Klein-Gordon/Dirac equations, Lagrangian field theory, Examples of quantum field theory − ϕ^4, Quantum electrodynamics.

5. **Elementary Particles**

   Elementary particles − weak and strong interactions − selection rules − CPT theorem − Symmetry Principles in Particle Physics.

Sample questions for Theoretical Physics

1. a) A mass M is constrained to slide without friction on a horizontal track AB. A mass M is connected to M by a massless inextensible string. Write down the Lagrangian of the system. Write down also the equation of motion. Making small angle approximation find the angular frequency of small oscillation.

   b) Consider a binary star system.

      i) Write down the Lagrangian of the system in terms of Cartesian coordinates of the two stars $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$.

ii) Show that the potential energy is a homogeneous function of the coordinates of degree -1 i.e.

$$V\left(\alpha\overrightarrow{r_1}, \alpha\overrightarrow{r_2}\right) = \alpha^{-1}V\left(\overrightarrow{r_1}, \overrightarrow{r_2}\right)$$

where α is a real scaling parameter.

2. A wire bent as a parabola $y = ax^2$ is located in a uniform magnetic field of induction B, the vector $\vec{B}$ being perpendicular to the plane (x, y). At time $t = 0$, a connector starts sliding translation wise from the vertex of the parabola with a constant acceleration f. Find the e.m.f. of electromagnetic induction in the loop thus formed as a function of y.

3. Consider a system of N non-interacting atoms at absolute temperature T. They are placed in an external magnetic field $\overrightarrow{H}$ pointing along z-direction. Each atom has a magnetic moment $\overrightarrow{\mu} = p(\hbar\overrightarrow{J})$where p is constant and $\hbar\overrightarrow{J}$ is the total angular momentum of each atom.

   Treat the problem in a quantum mechanical way.

   a) Find the partition function of a single atom.
   b) Find the expression for the z-component of the magnetic moment of a single atom, in terms of the above partition function.
   c) What will be the partition function of the total system?

4. Two identically charged spheres are suspended by strings of equal length. The strings make an angle of 30^o with each other. When suspended in a liquid of density $(800kg/m^3)$, the angle remains the same. What is the dielectric constant of the liquid? (The density of the material of the sphere is $1600kg/m^3$).

5. (a) A light beam is propagating through a block of glass with refractive index 1.5. If the block is moving with constant velocity in the same direction as the beam, what is the velocity of light in the block as measured by an observer in the laboratory?

   (b) A particle of mass $m_1 = 1gm$ travelling at a speed of $.9c$, c being the speed of light, collides head on with a stationary particle of mass $m_2 = 10gms$ and is embedded in it. What is the rest mass and velocity of the resulting composite particle?

6. A particle of mass m and charge e enters a homogeneous and stationary electric field E with velocity v perpendicular to the direction of the field. Calculate the particle's path.

7. (a) If the effective density of states in valence band is eight times that in convection band in a pure semiconductor at 27^oC, find the shift of Fermi level from the middle of the energy gap assuming low concentration of electrons and holes in the semicondctor. (Boltzmann's constant $K_B = 1.33 \times 10^{-23} J/\text{ mole}^o\ K$, In $2 = .693$)

(b) Consider a line of $2N$ ions of alternating charges $\pm q$ with a repulsive potential A/R^n between nearest neighbours in addition to the usual Coulomb potential. Neglecting surface effects find the equilibrium separation R_0 for such a system. Let the crystal be compressed so that R_0 becomes $R_0(1-\delta)$. Calculate work done in compressing a unit length of the crystal to order δ^2.

8. a) A hypothetical semi-conductor has a conduction band (cb) that can be described by $E_{cb} = E_1 - E_2 \cos(ka)$ and valence band (vb) which is represented by $E_{vb} = E_3 - E_4 \sin^2\left(\frac{ka}{2}\right)$ where $E_3 < (E_1 - E_2)$ and $-\pi/a < k < \pi/a$. Find out the expressions for

   i) the band-widths of the conduction band and the valence band,

   ii) the band-gap of the material,

   iii) the effective mass of the electrons at the bottom of the conduction band.

   (b) A beam of electrons with kinetic energy 1 keV is diffracted as it passes through a polycrystalline metal foil. The metal has a cubic crystal structure with a spacing of $1A^0$. Calculate the wave length of electron and the Bragg angle for the first order diffraction. Take m, h, c the mass of the electron, Planck's constant and speed of light respectively as follows: $mc^2 = .5$Mev, $c = 3 \times 10^8$m/s, $h = 6.6 \times 10^{-34}$Js. Also take $1eV = 1.6 \times 10^{-19}$J.

9. a) A spaceship is moving away from the earth at a speed $v = 0.8Cc$. When the ship is at a distance of 6.66×10^8 Km from the earth as measured from the earth's reference frame, a radio signal is sent out to the spaceship by an observer on earth. How long will it take for the signal to reach the ship;

   i) As measured in the ship's frame of reference ?

   ii) As measured in the earth's reference frame ?

   b) A mirror is moving through vacuum with relativistic speed v in the x-direction with respect to an observer. A beam of light with frequency ω_1 is normally incident (from $x = +\infty$) on the mirror.

   i) Express the frequency (as measured by the observer) of the reflected light in terms of ω_1, c and v, c being the velocity of light.

   ii) what is the energy of each reflected photon?

10. a) For a three dimensional Harmonic oscillator the spectrum is given by

$$E_{n_x,n_y,n_z} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$$

   Calculate the degeneracy factor.

b) Show that

$$e^{i\alpha\sigma_x}\ \sigma_z\ e^{-i\alpha\sigma_x} \ = \ \sigma_z \cos 2\alpha + \sigma_y \sin 2\alpha$$

where σ_x, σ_y, σ_z are Pauli matrices.

11. Consider the theory of Dirac fermions interacting with photons (QED).

 i) Find expression for the charge current density and show that it is conserved.

 ii) Using Wick's theorem, derive amplitudes for the following processes:

  (a) electron - positron annihilation process of $O(e^2)$ where e is the coupling constant.

  (b) Same process of $O(e^4)$.

  (c) Draw the Feynman diagrams of the above processes.

 iii) Consider the energy projection operators $\Lambda^{\pm}(p) = \pm\frac{\not{p} + mc}{2mc}$, where $\not{p} = \gamma_\mu p^\mu$. Prove the following identities :

  (a) $(\Lambda^+)^2 = \Lambda^+$

  (b) $\Lambda^+\Lambda^- = 0$

12. a) The following two decays are forbidden. State the reason;

  i) $p \to \pi^0 + e^+$

  ii) $n \to \pi^+ + \mu^-$

 The following two decays are allowed. Give justifications;

  i) $n \to p + e^- + \overline{\nu_e}$

  ii) $\pi^+ \to \mu^+ + \nu_\mu$

 b) Obtain the exact ratio of the energy released when 1 gm of Uranium undergoes fission to the energy released when 1 gm of TNT explodes.

13. A thin fixed ring of radius 1 metre has a positive charge of 10^{-5} coulomb uniformly distributed over it. A particle of mass .9 gm and having a negative charge of 10^{-6} coulomb is placed on the axis of the ring at a distance of 1 cm from the centre of the ring. Show that the motion of the particle is approximately simple harmonic. Find the time period of oscillation.

14. Show that the fields

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0 r^2}\ \theta(vt - r)\hat{r},\ \vec{B}(\vec{r},t) \ = \ 0$$

(where the step function $\theta(x) = 1$ if $x > 0$ and 0 otherwise), satisfy all the Maxwell's equations. Determine the corresponding ρ and $\vec{J}$. Describe the physical situation that gives rise to these fields.

PART-IV

Statistics
Syllabus

Probability and Sampling Distributions : Notion of sample space, combinatorial probability, conditional probability and independence, random variable and expectation, moments, standard discrete and continuous distributions, sampling distribution of statistics based on normal samples.

Descriptive Statistics (including Numerical Analysis) : Descriptive measures, graduation of frequency curves, correlation and regression (bivariate and multivariate), polynomial interpolation, numerical integration.

Inference : Elementary theory and methods of estimation (unbiasedness, minimum variance, sufficiency, mle, method of moments). Testing of hypotheses (basic concepts and simple applications of Neyman-Pearson lemma).

Designs (including elementary ANOVA) and Sample Surveys : Basic designs, (CRD/RBD/LSD) and their analysis, conventional sampling techniques (srswr/wor) including stratification.

Sample Questions

1. The standard deviation of two sets containing n_1 and n_2 members are σ_1 and σ_2 respectively, being measured from their respective means m_1 and m_2. If the two sets are grouped together as one set of (n_1+n_2) members, show that the standard deviation σ of this set measured from its mean is given by

$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1+n_2} + \frac{n_1 n_2}{(n_1+n_2)^2}(m_1-m_2)^2.$$

2. An unbiased coin is tossed $(m+n)$ times where $m \geq n$. Find the probability of getting at least m consecutive heads ; hence find the probability of getting exactly m consecutive heads.

3. a) Find the probability density function (p.d.f.) of X^2 where X has the uniform distribution over $(-2, 3)$.

   b) Find the p.d.f. of $(X+Y)$ where X and Y are independent, and X has the p.d.f.

$$f_X(x) \;=\; \begin{cases} 1-|1-x| & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   and Y has the uniform distribution over (0, 1).

4. Find the maximum likelihood estimator of θ for a random sample from a distribution having the p.d.f.

$$f_\theta(x) = \begin{cases} (\theta+1)x^\theta & \text{if } 0 \le x \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find also the estimator of θ by the method of moments.

5. Describe the model and analysis used in Randomised Block Design. Explain the utility of Latin Square Design. Construct a 4×4 Latin Square Design.

6. Let X_1 X_2 be two iid Bernoulli random variables with probability of success p. Find the most powerful test for testing $H_0 : p = \dfrac{1}{2}$ against $H_1 : p = \dfrac{2}{3}$ at level of significance $\alpha = .05$.

7. Let X, Y have the joint p.d.f.

$$f(x,y) = \begin{cases} \dfrac{6-x-y}{8} & \text{if } 0 \le x \le 2,\ 2 \le y \le 4; \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X+Y<3)$ and $P(X<1|Y=3)$.

8. Let $X_1, ..., X_n$ be *iid* each following $Bin(1;p)$ where p is unknown and $0 < p < 1$. Let α, β be two integers (≥ 1).

   a) Find a statistic T based on $\sum_{i=1}^{n} X_i$ such that T is an unbiased estimator of $p^\alpha(1-p)^\beta$ where $\alpha + \beta \le n$.

   b) Discuss the above problem in case $\alpha + \beta > n$.

9. To estimate the total number of workers employed in an industry comprising 90 factories in all, a sample of 10 factories are selected in the following manner : The smallest two factories are included in the sample and in addition a simple random sample (without replacement) of size 8 is selected from the remaining 88 factories. The estimator proposed is $90\bar{y}$ where $\bar{y}$ is the sample mean of number of workers of all the 10 factories in the sample.

   Show that the estimator is not unbiased. Suggest an unbiased estimator and prove that your estimator is unbiased.

10. Let X_1 and X_2 constitute a random sample of size 2 from the population with a density of the form $f(x|\theta) = \theta x^{\theta-1}$ for $0 < x < 1$.

    If the critical region "$x_1 x_2 \ge \frac{3}{4}$" is used to test the null hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$, what is the power of this test at $\theta = 2$?

11. Let the distribution of X under θ be given by

x	0	1	2	3
$P_\theta(X = x)$	θ	2θ	$0.9 - 2\theta$	$0.1 - \theta$

where $0 < \theta \leq 0.1$.

Determine (if any) which one of the following tests is UMP at level $\alpha = 0.05$ for testing $H_0 : \theta = 0.05$ vs. $H_1 : \theta > 0.05$. (Below $\phi(x)$ stands for the probability of rejecting H_0 when x is observed).

(i) $\phi(0) = 1, \quad \phi(1) = \phi(2) = \phi(3) = 0;$

(ii) $\phi(1) = 0.5, \quad \phi(0) = \phi(2) = \phi(3) = 0;$

(iii) $\phi(3) = 1, \quad \phi(0) = \phi(1) = \phi(2) = 0;$

12. In estimating the mean of a finite survey population on drawing a sample of a given size from it explain how one may involve an 'analysis of variance' arrgument to justify the efficicacy of stratified sampling. Develop a principle in formation of strata. If you are given a stratified simple random sample taken without replacement independently stratum-wise, show how you may derive an unbiased estimator for the population variance.

13. For the following two-way linear fixed-effect model,

$$E(y_{ij}) = \mu + \alpha_i + \beta_j, \quad \sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = 0,$$

show that

$$\sum_{ij} (y_{ij} - \mu - \alpha_i - \beta_j)^2$$

$$= pq(y_{00} - \mu)^2 + q \sum_i (y_{i0} - y_{00} - \alpha_i)^2$$

$$+p \sum_j (y_{0j} - y_{00} - \beta_j)^2 + \sum_{ij} (y_{ij} - y_{i0} - y_{0j} + y_{00})^2$$

where $y_{i0} = \dfrac{\sum_j y_{ij}}{q}, \quad y_{0j} = \dfrac{\sum_i y_{ij}}{p}, \quad y_{00} = \dfrac{\sum_{ij} y_{ij}}{pq}.$

Hence obtain the least-square estimates of μ, α_i and β_j. Use this identity to split the total sum of squares into three orthogonal sum of squares and identify them properly.

14. Based on a simple random sample of size 200, a 95% confidence interval for the population mean turned out to be (10, 20). Find a 90% confidence interval for the population mean based on this information.