JRF IN MATHEMATICS 2004-2005

There will be two tests RM-1, and RM-2 of 2 hours duration each in the forenoon and in the afternoon. Topics to be covered in these tests along with an outline of the syllabus and sample questions are given below:

1) Topics for MI (Forenoon examination) : Real Analysis, Measure and Integration, Complex Analysis, Ordinary Differential Equations and General Topology.

2) Topics for MII (Afternoon examination) : Algebra, Linear Algebra, Functional Analysis, Elementary Number Theory and Combinatorics.

Candidates will be judged based on their performance in **both** the tests.

OUTLINE OF THE SYLLABUS

1. **General Topology :** Topological spaces, Continuous functions, Connectedness, Compactness, Separation Axioms. Product spaces. Complete metric spaces. Uniform continuity.

2. **Functional Analysis :** Normed linear spaces, Banach spaces, Hilbert spaces, Compact operators. Knowledge of some standard examples like $C[0,1], L^p[0,1]$. Continuous linear maps (linear operators). Hahn-Banach Theorem, Open mapping theorem, Closed graph theorem and the uniform boundedness principle.

3. **Real analysis :** Sequences and series, Continuity and differentiability of real valued functions of one and two real variables and applications, uniform convergence, Riemann integration.

4. **Linear algebra :** Vector spaces, linear transformations, characteristic roots and characteristic vectors, systems of linear equations, inner product spaces, diagonalization of symmetric and Hermitian matrices, quadratic forms.

5. **Elementary number theory :** Divisibility, congruences, standard arithmetic functions, permutations and combinations.

6. **Lebesgue integration :** Lebesgue measure on the line, measurable functions, Lebesgue integral, convergence almost everywhere, monotone and dominated convergence theorems.

7. **Complex analysis :** Analytic functions, Cauchy's theorem and Cauchy integral formula, maximum modulus principle, Laurent series, Singularities, Theory of residues, contour integration.

8. **Abstract algebra :** Groups, Symmetric and Alternating groups, Direct product and finite abelian groups, Sylow theorems; rings, polynomial rings, integral domains, Euclidean rings; fields, extension fields, roots of polyno-

mials, finite fields.

9. **Ordinary differential equations :** First order ODE and their solutions, singular solutions, initial value problems for first order ODE, general theory of homogeneous and nonhomogeneous linear differential equations.

SAMPLE QUESTIONS

1. Let $p(x)$ be an odd degree polynomial in one variable with coefficients from the set R of real numbers. Let $g : R \to R$ be a bounded continuous function. Prove that there exists an $x_0 \in R$ such that $p(x_0) = g(x_0)$.

2. Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. Define

$$s_n = \frac{(a_1 + a_2 + \ldots + a_n)}{n}, n = 1, 2, 3, \ldots$$

Show that $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} s_n$.

3. Let f be an analytic function defined on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. If $|f(z)| \leq 1 - |z|$ for each $z \in D$, then show that f is the zero function on D.

4. Let ω be a primitive nth root of unity. Show that

$$(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1}) = n.$$

[Hint: Consider the polynomial $z^n - 1$].
Deduce that if $A_1, A_2, \ldots, A_n$ are the vertices of a regular n-gon inscribed in the unit circle, then $\ell(A_1A_2)\ldots\ell(A_{n-1}A_n) = n$.

5. If a, b, c are real numbers, then show that

$$(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 \geq (ab+bc+ca).$$

6. Let λ denote the Lebesque measure on [0,1]. Define the intervals $I_{n,i} = \left[\frac{2i}{2n}, \frac{2i+1}{2n}\right)$ and $J_{n,i} = \left[\frac{2i+1}{2n}, \frac{2i+2}{2n}\right), i = 0, 1, \ldots, n-1$. Consider the sequence of functions $f_1, f_2, \ldots$ on [0,1) defined by

$$f_n(x) = \begin{cases} 1 & \text{if} \quad x \in I_{n,i} \text{ for some } i \\ -1 & \text{if} \quad x \in J_{n,i} \text{ for some } i. \end{cases}$$

Show that

$$\lim_{n\to\infty} \int_a^b f_n d\lambda = 0$$

for every interval $[a, b]$ contained in (0,1).

7. Let f be a uniformly continuous real valued function on the real line R. Assume that f is integrable with respect to the Lebesgue measure on R. Show that $f(x) \to 0$ as $|x| \to \infty$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the differential equation

$$y'(t) + y(t) = f(t) \qquad (*)$$

on $\mathbb{R}$.
a) Show that $(*)$ can have atmost one bounded solution.
b) If f is bounded, show that $(*)$ has a bounded solution.

9. Let $q(X)$ be a polynomial in X of degree n with real coefficients and let k be a non-zero real number. Show that the differential equation

$$\frac{dy}{dx} + ky(x) = q(x)$$

has exactly one polynomial solution of degree n.

10. Let X be a Hausdorff space. Let $f : X \to \mathbb{R}$ be such that $\{(x, f(x)) : x \in X\}$ is a compact subset of $X \times \mathbb{R}$. Show that f is continuous.

11. Let X be a compact Hausdorff space. Assume that the vector space of real-valued continuous functions on X is finite dimensional. Show that X is finite.

12. Let (X, d) be a complete metric space, $A_1 \supseteq A_2 \supseteq \ldots$ be a sequence of closed sets in X such that $\sup\{d(x, y) : x, y \in A_n\}$ tends to zero as n tends to infinity. Let $f : X \to X$ be a continuous map. Show that

$$f(\bigcap_n A_n) = \bigcap_n f(A_n).$$

13. Let I be the ideal generated by $X - 3$ and 7 in the polynomial ring $\mathbb{Z}[X]$. Show that, for each $f(X) \in \mathbb{Z}[X]$, there exists a unique integer $a, 0 \leq a \leq 6$, such that $f(X) - a \in I$.

14. Show that there is no field isomorphism between $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$. Are they isomorphic as vector spaces over $\mathbb{Q}$?

15. Determine finite subgroups of the miltiplicative group of non-zero complex numbers.

16. Let L and T be two linear transformations from a real vector space V to $\mathbb{R}$ such that $L(v) = 0$ implies $T(v) = 0$. Show that $T = cL$ for some real number c.

17. Let

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Show that, for each nonzero scalar $\lambda, (\lambda I - B)^{-1} = P_\lambda(B)$ for some polynomial $P_\lambda(X)$ of degree 3.

18. Let X and Y be complex, normed linear spaces which are not necessarily complete. Let $T : X \to Y$ be a linear map such that $\{Tx_n\}$ is a Cauchy sequence in Y whenever $\{x_n\}$ is a Cauchy sequence in X. Show that T is continuous.

19. Let p be a prime and r an integer, $0 < r < p$. Show that $\frac{(p-1)!}{r!(p-r)!}$ is an integer.

20. If a and b are integers such that 9 divides $a^2 + ab + b^2$ then show that 3 divides both a and b.

21. Let S_n denote the group of permutations of $\{1, 2, 3, \ldots, n\}$ and let k be an integer between 1 and n. Find the number of elements x in S_n

such that the cycle containing 1 in the cycle decomposition of x has length k.