Syllabus for the M. Math. Selection Test–2010
TEST CODE MM

Open sets, closed sets and compact sets in $\mathbf{R}^n$;

Convergence and divergence of sequences and series;

Continuity, uniform continuity, differentiability, mean-value theorem;

Pointwise and uniform convergence of sequences and series of functions, Taylor expansions, power series;

Integral calculus of one variable : Riemann integration, Fundamental theorem of calculus, change of variables;

Directional and total derivatives, Jacobians, chain rule;

Maxima and minima of functions of one and several variables;

Elementary topological notions for metric spaces : compactnes, connectedness, completeness;

Elements of ordinary differential equations.

Equivalence relations and partitions;

Primes and divisibility;

Groups : subgroups, products, quotients, homomorphisms, Lagrange's theorem, Sylow's theorems;

Commutative rings : Ideals, prime and maximal ideals, quotients, congruence arithmetic, integral domains, field of fractions, principal ideal domains, unique factorization domains, polynomial rings;

Fields : field extensions, roots and factorization of polynomials, finite fields;

Vector spaces: subspaces, basis, dimension, direct sum, quotient spaces;

Matrices : systems of linear equations, determinants, eigenvalues and eigenvectors, diagonalization, triangular forms;

Linear transformations and their representation as matrices, kernel and image, rank;

Inner product spaces, orthogonality and quadratic forms, conics and quadrics.

Sample Questions for the Selection Test

Notation : $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers, rational numbers, integers and natural numbers respectively.

(1) Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n\geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n\to\infty} f(x_n)$ exists.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \max\{| x |, | y |\}.$$

Show that f is a uniformly continuous function.

(3) A map $f : \mathbb{R} \to \mathbb{R}$ is called open if $f(A)$ is open for every open subset A of $\mathbb{R}$. Show that every continuous open map of $\mathbb{R}$ into itself is monotonic.

(4) Let $S = \{(x_1, x_2, ...x_n) \in \mathbb{R}^n : \sum | x_i |^2 = 1\}$. Let

$$A = \{(y_1, y_2, ...y_n) \in \mathbb{R}^n : \sum \frac{y_i}{i} = 0\}.$$

Show that the set $S + A = \{x + y : x \in S \ , \ y \in A\}$ is a closed subset of $\mathbb{R}^n$.

(5) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $f : [0,1] \to \mathbb{C}$ be continuous with $f(0) = 0,\ f(1) = 2$. Show that there exists at least one t_0 in $[0,1]$ such that $f(t_0)$ is in $\mathbb{T}$.

(6) Let f be a continuous function on $[0,1]$. Evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x) dx$$

(7) Let $N > 0$ and let $f : [0,1] \to [0,1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x. Show that f is Riemann integrable.

(8) Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational ,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ with } m, n \text{ relatively prime} \end{cases}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x > \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Show that $g \circ f$ is not Riemann integrable.

(9) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$. Define

$$f_n(x) = f(nx), \text{ for } x \in \mathbb{R} \text{ and } n = 1, 2, 3, \ldots$$

Suppose that $\{f_n\}$ is equicontinuous on $[0, 1]$, that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, y \in [0, 1]$, $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \varepsilon$ for all n. Show that $f(x) = 0$ for all $x \in [0, 1]$.

(10) Find the most general curve in $\mathbb{R}^2$ whose normal at each point passes through $(0, 0)$. Find the particular curve through $(2, 3)$.

(11) Find the maximum value of the function

$$f(x, y, z) = s(s - x)(s - y)(s - z),$$

where $s > 0$ is a given constant under the condition

$$x + y + z - 2s = 0,$$

and where x, y, z are restricted by the inequalities

$$x \geq 0, y \geq 0, z \geq 0,$$

$$x + y \geq z, x + z \geq y, y + z \geq x.$$

(12) Let (X, d) be a compact metric space and $f : X \to X$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is onto.

(13) Let ω be an n-th root of unity such that $\omega^m \neq 1$ for any positive integer $m < n$. Show that $(1 - \omega)...(1 - \omega^{n-1}) = n$ [Hint : Consider the polynomial $z^n - 1$].

Hence deduce the following : if $A_1, A_2, ..., A_n$ are the vertices of a regular n-gon inscribed in a unit circle, prove that

$$l(A_1A_2)l(A_1A_3)...l(A_1A_n) = n,$$

where $l(AB)$ denotes the length of a line segment AB.

(14) Let $f(x)$ be a non-constant polynomial with integer coefficients. Show that the set $S = \{f(n) | n \in \mathbb{N}\}$ has infinitely many composite numbers.

(15) Let G be any group. Prove that any subgroup H of finite index n in G contains a normal subgroup of index dividing $n!$.
Hint : Consider the homomorphism from G to the group of permutations of the set of left cosets of H in G.

(16) Let G be a nonabelian group of order 55. How many subgroups of order 11 does it have? Using this information or otherwise compute the number of subgroups of order 5.

(17) Let $n \in \mathbb{N}$ and p be a prime number. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_\ell x^\ell$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$, where $a_i, b_j \in \mathbb{Z}/p^n\mathbb{Z}$, for all $0 \le i \le \ell$, $0 \le j \le m$. Suppose that $fg = 0$. Prove that $a_ib_j = 0$ for all $0 \le i \le \ell$, $0 \le j \le m$.

(18) Let $a_1, a_2, ..., a_n$ be n distinct integers. Prove that the polynomial $f(x) = (x - a_1)(x - a_2)...(x - a_n) + 1$ is irreducible in $\mathbb{Z}[x]$.

(19) Prove that $x^4 - 10x^2 + 1$ is reducible modulo p for every prime p.

(20) Consider the two fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$, where $\mathbb{Q}$ is the field of rational numbers. Show that they are isomorphic as vector spaces but not isomorphic as fields.

(21) Show that the only field automorphism of $\mathbb{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbb{R}$ is the identity.

(22) Suppose $f \in F[x]$ be an irreducible polynomial of degree 5, where F is a field. Let K be a quadratic field extension of F, that is, $[K : F] = 2$. Prove that f remains irreducible over K.

(23) Let $k[x, y]$ be the polynomial ring in two variables x and y over a field k. Prove that any ideal of the form $I = (x - a, y - b)$ for $a, b \in k$ is a maximal ideal of this ring. What is the vector space dimension (over k) of the quotient space $k[x, y]/I$?

(24) Let A be a $n \times n$ symmetric matrix of rank 1 over the complex numbers $\mathbb{C}$. Show that $A = \alpha\, \boldsymbol{u}\boldsymbol{u}^t$ for some non-zero scalar

$\alpha \in \mathbb{C}$ and a non-zero vector $\boldsymbol{u} \in \mathbb{C}^n$ (where $\boldsymbol{u}^t$ is the transpose of $\boldsymbol{u}$).

(25) Let A be any 2×2 matrix over $\mathbb{C}$ and let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be any polynomial over $\mathbb{C}$. Show that $f(A)$ is a matrix which can be written as $c_0 I + c_1 A$ for some $c_0, c_1 \in \mathbb{C}$, where I is the identity matrix.

(26) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that there is a line L through origin such that $T(L) = L$

(27) Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0$ for all $(i, j) \neq (1, 2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.

Model question paper

Time : 2 hours

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a): You need to answer 8 questions.
b): Attempt any 4 questions from each group.
c): Each question carries 10 marks:  Total Marks = 80.
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A. Analysis

(1) Let $f_1(x) = 1, f_2(x) = e^x, f_3(x) = \sin x$ for $0 \leq x \leq 1$. Show that f_1, f_2, f_3 are linearly independent as elements of the vector space of continuous functions $C[0, 1]$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A real number $\delta \neq 0$ is said to be a period of f if $f(x + \delta) = f(x)$ for all $x \in \mathbb{R}$. Let

$$\alpha = \inf\{\delta > 0 \mid \delta \text{ is a period of } f\}.$$

Suppose $\alpha \neq 0$ and that α is a period of f. Show that any period of f is an integral multiple of α.

(3) Show that

$$\sup_{a>b>0} \left(\frac{a-b}{\sqrt{a}}\right)\left(\frac{1}{\sqrt{a}-\sqrt{b}}\right) = 2.$$

Use this to show that

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{a_n + a_{n+1} + \dots}} \leq 2\sqrt{\sum_{n=1}^{\infty} a_n}$$

for any sequence of positive numbers $\{a_n\}$.

(4) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Assume that

$$\lim_{x\to\infty} \frac{f(x)}{x_m} = 1$$

where $m \geq 1$ is an integer. Show that

$$\lim_{x\to\infty} \frac{m+1}{x^{m+1}} \int_0^x f(y)dy = 1.$$

(5) Let $f : [0;1] \to \mathbb{R}$ be a function such that the sequence $\{f(a_n)\}$ is Cauchy whenever $\{a_n\}$ is a Cauchy sequence in $[0,1]$. Show that f is continuous.

(6) Let $a \geq 0$ be a constant. Consider the differential equation

$$tx''(t) + ax'(t) = 0, t > 1$$

with the boundary condition $x(1) = 1$. Show that a unique bounded solution exists if and only if $a \leq 1$.

B. Algebra

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T^2 = 0$. Show that the rank of T does not exceed $n/2$.

(2) Let A be a symmetric 3×3 matrix with non negative entries in which each row adds up to 1. Show that 1 is an eigenvalue of A and find a corresponding eigenvector (x_1, x_2, x_3) with $x_i > 0$ for all i.

(3) Let A be a non zero $n \times n$ matrix with complex entries such that $A^3 = 0$. Let I denote the $n \times n$ identity matrix. Find the eigenvalues of the matrix $I + A$.

(4) Let G denote the group of all complex numbers of modulus 1. Is G isomorphic to $G \times G$? Justify.

(5) Count the number of elements of order 50 in a cyclic group of order 550.

(6) Let $\mathbb{C}[x, y]$ denote the ring of polynomials in x and y with coefficients in the field $\mathbb{C}$ of complex numbers. Let I be the ideal generated by the polynomials $x^2 - y^3$ and xy. Show that I is not a prime ideal in $\mathbb{C}[x, y]$.