Test Code: CS (Short answer type) 2010

M.Tech. in Computer Science

The candidates for M.Tech. in Computer Science will have to take two tests – Test MIII (objective type) in the forenoon session and Test CS (short answer type) in the afternoon session. The CS test booklet will have two groups as follows.

GROUP A

A test for all candidates in analytical ability and mathematics at the B.Sc. (pass) level, carrying 28 marks.

GROUP B

A test, divided into several sections, carrying equal marks of 72 in mathematics, statistics, and physics at the B. Sc. (Hons.) level, and in computer science, and engineering and technology at the B.Tech. level. A candidate has to answer questions from ***only one*** of these sections according to his/her choice.

The syllabi and sample questions for the CS test are given below.

Note: Not all questions in the sample set are of equal difficulty. They may not carry equal marks in the test.

Syllabus

GROUP A

Elements of set theory. Permutations and combinations. Functions and relations. Theory of equations. Inequalities.
Limits, continuity, sequences and series, differentiation and integration with applications, maxima-minima, complex numbers and De Moivre's theorem.
Elementary Euclidean geometry and trigonometry.
Elementary number theory, divisibility, congruences, primality.
Determinants, matrices, solutions of linear equations, vector spaces, linear independence, dimension, rank and inverse.
Logical reasoning.

GROUP B

Mathematics
(B.Sc. Hons. level)

In addition to the syllabus for Mathematics in Group A, the syllabus includes:

Calculus and real analysis – real numbers, basic properties; convergence of sequences and series; limits, continuity, uniform continuity of functions; differentiability of functions of one or more variables and applications. Indefinite integral, fundamental theorem of Calculus, Riemann integration, improper integrals, double and multiple integrals and applications. Sequences and series of functions, uniform convergence.
Linear algebra – vector spaces and linear transformations; matrices and systems of linear equations, characteristic roots and characteristic vectors, Cayley-Hamilton theorem, canonical forms, quadratic forms.
Graph Theory – connectedness, trees, vertex coloring, planar graphs, Eulerian graphs, Hamiltonian graphs, digraphs and tournaments.
Abstract algebra – groups, subgroups, cosets, Lagrange's theorem; normal subgroups and quotient groups; permutation groups; rings, subrings, ideals, integral domains, fields, characteristics of a field, polynomial rings, unique factorization domains, field extensions, finite fields.
Differential equations – solutions of ordinary and partial differential equations and applications.

Statistics
(B.Sc. Hons. level)

Notions of sample space and probability, combinatorial probability, conditional probability, Bayes' theorem and independence, random variable and expectation, moments, standard univariate discrete and continuous distributions, sampling distribution of statistics based on normal samples, central limit theorem, approximation of binomial to normal. Poisson law, multinomial, bivariate normal and multivariate normal distributions.

Descriptive statistical measures, product-moment correlation, partial and multiple correlation; regression (simple and multiple); elementary theory and methods of estimation (unbiasedness, minimum variance, sufficiency, maximum likelihood method, method of moments, least squares methods). Tests of hypotheses (basic concepts and simple applications of Neyman-Pearson Lemma). Confidence intervals. Tests of regression. Elements of non-parametric inference. Contingency tables and Chi-square, ANOVA, basic designs (CRD/RBD/LSD) and their analyses. Elements of factorial designs. Conventional sampling techniques, ratio and regression methods of estimation.

Physics
(B.Sc. Hons. level)

General properties of matter – elasticity, surface tension, viscosity.

Classical dynamics – Lagrangian and Hamiltonian formulation, symmetries and conservation laws, motion in central field of force, planetary motion, collision and scattering, mechanics of system of particles, small oscillation and normal modes, wave motion, special theory of relativity.

Electrodynamics – electrostatics, magnetostatics, electromagnetic induction, self and mutual inductance, capacitance, Maxwell's equation in free space and linear isotropic media, boundary conditions of fields at interfaces.

Nonrelativistic quantum mechanics – Planck's law, photoelectric effect, Compton effect, wave-particle duality, Heisenberg's uncertainty principle, quantum mechanics, Schrodinger's equation, and some applications.

Thermodynamics and statistical Physics – laws of thermodynamics and their consequences, thermodynamic potentials and Maxwell's relations, chemical potential, phase equilibrium, phase space, microstates and macrostates, partition function free energy, classical and quantum statistics.

Electronics – semiconductor physics, diode as a circuit element, clipping, clamping, rectification, Zener regulated power supply, transistor as a circuit element, CC CB CE configuration, transistor as a switch, OR and NOT gates feedback in amplifiers.

Operational Amplifier and its applications – inverting, noninverting amplifiers, adder, integrator, differentiator, waveform generator comparator and Schmidt trigger.

Digital integrated circuits – NAND, NOR gates as building blocks, XOR gates, combinational circuits, half and full adder.

Atomic and molecular physics – quantum states of an electron in an atom, Hydrogen atom spectrum, electron spin, spin–orbit coupling, fine structure, Zeeman effect, lasers.
Condensed matter physics – crystal classes, 2D and 3D lattice, reciprocal lattice, bonding, diffraction and structure factor, point defects and dislocations, lattice vibration, free electron theory, electron motion in periodic potential, energy bands in metals, insulators and semiconductors, Hall effect, thermoelectric power, electron transport in semiconductors, dielectrics, Claussius Mossotti equation, Piezo, pyro and ferro electricity.
Nuclear and particle physics – Basics of nuclear properties, nuclear forces, nuclear structures, nuclear reactions, interaction of charged particles and e-m rays with matter, theoretical understanding of radioactive decay, particle physics at the elementary level.

Computer Science
(B.Tech. level)

Data structures - array, stack, queue, linked list, binary tree, heap, AVL tree, B-tree.
Programming languages - Fundamental concepts – abstract data types, procedure call and parameter passing, languages like C and C++.
Design and analysis of algorithms – Asymptotic notation, sorting, selection, searching.
Computer organization and architecture - Number representation, computer arithmetic, memory organization, I/O organization, microprogramming, pipelining, instruction level parallelism.
Operating systems - Memory management, processor management, critical section problem, deadlocks, device management, file systems.
Formal languages and automata theory - Finite automata and regular expressions, pushdown automata, context-free grammars, Turing machines, elements of undecidability.
Principles of Compiler Construction - Lexical analyzer, parser, syntax-directed translation, intermediate code generation.
Database management systems - Relational model, relational algebra, relational calculus, functional dependency, normalization (up to 3rd normal form).
Computer networks - OSI, LAN technology - Bus/tree, Ring, Star; MAC protocols; WAN technology - circuit switching, packet switching; data communications - data encoding, routing, flow control, error detection/correction, Internetworking, TCP/IP networking including IPv4.

Switching Theory and Logic Design - Boolean algebra, minimization of Boolean functions, combinational and sequential circuits – synthesis and design.

Engineering and Technology
(B.Tech. level)

Moments of inertia, motion of a particle in two dimensions, elasticity, friction, strength of materials, surface tension, viscosity and gravitation.
Laws of thermodynamics, and heat engines, optics.
Electrostatics, magnetostatics and electromagnetic induction.
Magnetic properties of matter - dia, para and ferromagnetism.
Laws of electrical circuits - RC, RL and RLC circuits, measurement of current, voltage and resistance.
D.C. generators, D.C. motors, induction motors, alternators, transformers, single phase machine.
p-n junction, bipolar & FET devices, transistor amplifier, oscillator, multi-vibrator, operational amplifier.
Digital circuits - logic gates, multiplexer, de-multiplexer, counter, A/D and D/A converters.
Boolean algebra, minimization of switching functions, combinational and sequential circuits.
C Programming language.

Sample Questions

GROUP A

Mathematics

A1. If 1, $a_1, a_2, \ldots, a_{n-1}$ are the n roots of unity, find the value of

$$(1 - a_1)(1 - a_2)\ldots(1 - a_{n-1}).$$

A2. Let

$S = \{(a_1, a_2, a_3, a_4) : a_i \in \Re, i = 1, 2, 3, 4 \text{ and } a_1 + a_2 + a_3 + a_4 = 0\}$

and

$\Gamma = \{(a_1, a_2, a_3, a_4) : a_i \in \Re, i = 1, 2, 3, 4 \text{ and } a_1 - a_2 + a_3 - a_4 = 0\}$.

Find a basis for $S \cap \Gamma$.

A3. Provide the inverse of the following matrix:

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_2 & c_3 & c_0 & c_1 \\ c_3 & -c_2 & c_1 & -c_0 \\ c_1 & -c_0 & c_3 & -c_2 \end{pmatrix}$$

Where $c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$, and $c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$.

(Hint: What is $c_0^2 + c_1^2 + c_2^2 + c_3^2$?)

A4. For any real number x and for any positive integer n show that

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$$

where $[a]$ denotes the largest integer less than or equal to a.

A5. Let $b_q b_{q-1} \ldots b_1 b_0$ be the binary representation of an integer b, i.e.,

$$b = \sum_{j=0}^{q} 2^j b_j, \; b_j = 0 \text{ or } 1, \text{ for } j = 0, 1, \ldots, q.$$

Show that b is divisible by 3 if $b_0 - b_1 + b_2 - \ldots + (-1)^q b_q = 0$.

A6. A sequence $\{x_n\}$ is defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$, $n = 1, 2, \ldots$
Show that the sequence converges and find its limit.

A7. Find the following limit:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right)$$

A8. Find the total number of English words (all of which may not have proper English meaning) of length 10, where all ten letters in a word are not distinct.

A9. Let $a_0 + \frac{a_1}{2}+\frac{a_2}{3}+\ldots..+\frac{a_n}{n+1}=0,$ where a_i's are some real constants. Prove that the equation $a_0+a_1x+a_2x^2+\ldots+a_nx^n=0$ has at least one solution in the interval (0, 1).

A10. Let $\phi(n)$ be the number of positive integers less than n and having no common factor with n. For example, for $n = 8$, the numbers 1, 3, 5, 7 have no common factors with 8, and hence $\phi(8) = 4$. Show that

(i) $\phi(p)=p-1$,

(ii) $\phi(pq)=\phi(p)\phi(q)$, where p and q are prime numbers.

A11. Let T_n be the number of strings of length n formed by the characters a, b and c that do not contain cc as a substring.

(a) Find the value of T_4.

(b) Prove that $T_n \geq 2^{n+1}$ for $n > 1$.

A12. Let f be a real-valued function such that $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbf{R}$. Define a function ϕ by $\phi(x) = c + f(x)$, $x \in \mathbf{R}$, where c is a real constant. Show that for every positive integer n,

$$\phi^n(x)=(c+f(c)+f^2(c)+\ldots..+f^{n-1}(c))+f^n(x);$$

where, for a real-valued function g, $g^n(x)$ is defined by

$$g^0(x)=0,\ g^1(x)=g(x),\ g^{k+1}(x)=g(g^k(x))$$

A13. Consider a square grazing field with each side of length 8 metres. There is a pillar at the centre of the field (i.e. at the intersection of the two diagonals). A cow is tied to the pillar using a rope of length $\frac{8}{\sqrt{3}}$ meters. Find the area of the part of the field that the cow is allowed to graze.

A14. Let $f: [0,1] \to [-1,1]$ be such that $f(0) = 0$ and $f(x) = \sin\frac{1}{x}$ for $x > 0$. Is it possible to get three sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfying all the three properties P1, P2 and P3 stated below? If so, provide an example sequence for each of the three sequences. Otherwise, prove that it is impossible to get three such sequences.

P1: $a_n > 0$, $b_n > 0$, $c_n > 0$, for all n.

P2: $\lim_{n\to\infty} a_n = 0,\ \lim_{n\to\infty} b_n = 0,\ \lim_{n\to\infty} c_n = 0.$

P3: $\lim_{n\to\infty} f(a_n) = 0,\ \lim_{n\to\infty} f(b_n) = 0.5,\ \lim_{n\to\infty} f(c_n) = 1.$

A15. Let $a_1\ a_2\ a_3 \ldots\ a_k$ be the decimal representation of an integer a ($a_i \in \{0,\ldots,9\}$ for $i = 1,2,\ldots,k$). For example, if $a = 1031$, then $a_1=1$, $a_2=0$, $a_3=3$, $a_4=1$. Show that a is divisible by 11 if and only if

$$\sum_{\text{i odd}} a_i - \sum_{\text{i even}} a_i$$

is divisible by 11.

A16. Let $a < b < c < d$ be four real numbers, such that all six pairwise sums are distinct. The values of the smallest four pairwise sums are 1, 2, 3, and 4 respectively. What are the possible values of d? Justify your answer.

A17. Consider the 5×10 matrix A as given below.

$$A = \begin{pmatrix} 1 & 6 & 11 & 16 & 21 & 26 & 31 & 36 & 41 & 46 \\ 2 & 7 & 12 & 17 & 22 & 27 & 32 & 37 & 42 & 47 \\ 3 & 8 & 13 & 18 & 23 & 28 & 33 & 38 & 43 & 48 \\ 4 & 9 & 14 & 19 & 24 & 29 & 34 & 39 & 44 & 49 \\ 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \end{pmatrix}$$

Let a set of ten distinct elements $b_1, b_2, \ldots, b_{10}$ be chosen from A such that exactly two elements are chosen from each row and exactly one from each column. Show that $b_1 + b_2 + \ldots + b_{10}$ is always equal to 255.

GROUP B

Mathematics

M1. Let $0 < x_1 < 1$. If $x_{n+1} = \dfrac{x_n+3}{3x_n+1}$, $n = 1,2,3, \ldots$

(i) Show that $x_{n+2} = \dfrac{5x_n+3}{3x_n+5}$, $n = 1,2,3, \ldots$

(ii) Hence or otherwise, show that $\lim_{n\to\infty} x_n$ exists.

(iii) Find $\lim_{n\to\infty} x_n$.

M2. (a) A function f is defined over the real line as follows:

$$f(x) = \begin{cases} x\sin\frac{\pi}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Show that $f'(x)$ vanishes at infinitely many points in (0,1).

(b) Let $f:[0,1] \to \Re$ be a continuous function with $f(0) = 0$. Assume that f' is finite and increasing on (0,1).

Let $g(x) = \frac{f(x)}{x}$ $\quad x \in (0,1)$. Show that g is increasing.

M3. Let

$$f(x) = \begin{cases} (x-1) \quad (x^4 + 4x + 7) & \text{if } x \text{ is rational.} \\ (1-x) \quad (x^4 + 4x + 7) & \text{if } x \text{ is irrational.} \end{cases}$$

Find all the continuity points of f.

M4. Let h be any fixed positive real number. Show that there is no differentiable function $f:\Re \to \Re$ satisfying both the following conditions:

(a) $f'(0) = 0$.

(b) $f'(x) > h$ for all $x > 0$.

M5. Find the volume of the solid given by $0 \le y \le 2x$, $x^2 + y^2 \le 4$ and $0 \le z \le x$.

M6. (a) Let A, B and C be $1\times n$, $n\times n$ and $n\times 1$ matrices respectively. Prove or disprove: Rank$(ABC) \leq$ Rank(AC).

(b) Let S be the subspace of $\boldsymbol{R}^4$ defined by
$S = \{(a_1, a_2, a_3, a_4) : 5a_1 - 2a_3 - 3a_4 = 0\}$.
Find a basis for S.

M7. (a) A rumour spreads through a population of 5000 people at a rate proportional to the product of the number of people who have heard it and the number who have not. Suppose that 100 people initiate a rumour and that a total of 500 people know the rumour after two days. How long will it take for half the people to hear the rumour? [assume that $\frac{\log 9}{\log 49} = \frac{129}{229}$]

(b) Find the equation of the curve satisfying the differential equation

$$\frac{d^2y}{dx^2}(x^2+1) = 2x\frac{dy}{dx}.$$

M8. (a) Let $\{a_n : n \geq 1\}$ be a sequence of positive numbers. Define $b_n = \sqrt{a_{2n-1}a_{2n}}$ for $n \geq 1$. If a_n is monotonic and $\sum b_n$ converges, prove that $\sum a_n$ also converges.

(b) Let M be the set of all 3×3 matrices of the following for:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{pmatrix}$$

where a, b, $c \in Z_2$. Show that with standard matrix addition and multiplication (over Z_2), M is a commutative ring. Find all the idempotent elements of M.

M9. Consider the vector space of all n x n matrices over $\Re$.

(a) Show that there is a basis consisting of only symmetric and skew-symmetric matrices.

(b) Find out the number of skew-symmetric matrices this basis must contain.

M10. (a) Let G be a group. For a, b in G we say that b is *conjugate* to a (written $b \sim a$), if there exists g in G such that $b = gag^{-1}$. Show that $\sim$ is an equivalence relation on G. The equivalence classes of $\sim$ are called the *conjugacy classes* of G. Show that a subgroup N of G is normal in G if and only if N is a union of conjugacy classes.

(b) Let G be a group with no proper subgroups. Show that G is finite. Hence or otherwise, show that G is cyclic.

M11. Let V denote the vector space $\Re^n$. Suppose $V_n \to \Re$ is a function satisfying

- $f(v_1, v_2, ..., v_n) = 0$ whenever $v_i = v_j$ for some $i \neq j$
- $f(v_1, ..., v_{i-1}, \alpha v_i, v_{i+1}, ..., v_n) = \alpha f(v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n) \forall \alpha \in \Re$
- $f(v_1, ..., v_{i-1}, v_i + u_i, v_{i+1}, ..., v_n) = f(v_1, ..., v_{i-1}, v_i, v_{i+1}, ..., v_n) +$ $f(v_1, ..., v_{i-1}, u_i, v_{i+1}, ..., v_n) \forall u_i \in \Re^n$
- $f(e_1, ..., e_n) = 1$ where $e_1 = (1,0,...,0), e_2 = (0,1,0,...,0), ..., e_n = (0,...,0,1)$.

Show that for any $n \times n$ matrix A, whose columns are $v_1, v_2, ..., v_n$, $f(v_1, v_2, ..., v_n) = \det(A)$.

M12. (a) Consider the differential equation:

$$\frac{d^2 y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2 \cos^5 x.$$

By a suitable transformation, reduce this equation to a second order linear differential equation with constant coefficients. Hence or otherwise solve the equation.

(b) Find the surfaces whose tangent planes all pass through the origin.

M13. (a) Draw a simple graph with the degree sequence (1,1,1,1,4).
(b) Write down the adjacency matrix of the graph.
(c) Find the rank of the above matrix.
(d) Using definitions of characteristic root and characteristic vectors only, find out all the characteristic roots of the matrix in (b).

M14. Let A be any $n \times n$ real symmetric positive definite matrix. Let λ be the largest eigenvalue of A.

(a) Show that $\| Ax \| \leq \lambda x, \quad \forall \| x \| \neq 0$.

(b) Find $Sup_{\|x\| \neq 0} \dfrac{\| Ax \|}{\| x \|}$.

M15. Let $G = (V, E)$ be a connected simple graph. Our objective is to assign a direction to every edge, such that each node has in-degree at least one.

(a) Prove that such an assignment of directions is not possible if G is a tree.

(b) Prove that such an assignment of directions is always possible if G is not a tree.

Statistics

S1. (a) X and Y are two independent and identically distributed random variables with Prob[$X = i$] = p_i, for i = 0, 1, 2, ……… Find Prob[$X < Y$] in terms of the p_i values.

(b) Based on one random observation X from N(0, σ^2), show that $\sqrt{\pi/2}\,|X|$ is an unbiased estimate of σ.

S2. (a) Let X_0, X_1, X_2, … be independent and identically distributed random variables with common probability density function f. A random variable N is defined as

$$N = n \text{ if } \quad X_1 \leq X_0, \quad X_2 \leq X_0, \cdots, \quad X_{n-1} \leq X_0, \quad X_n > X_0, \quad n = 1, 2, 3, \cdots$$

Find the probability of $N = n$.

(b) Let X and Y be independent random variables distributed uniformly over the interval [0,1]. What is the probability that the integer closest to $\frac{X}{Y}$ is 2?

S3. Let A = {1,2,3}. You are given a coin with probability of head as p, where $0 < p < 1$ and p is unknown. Suggest a procedure for choosing a number randomly from A using the given coin, such that $P(\{1\}) = P(\{2\}) = P(\{3\}) = \frac{1}{3}$. Justify your answer.

S4. (a) Let $X_1, \dots, X_n$ be the iid $U(\theta-1, \theta+1)$ where θ is an unknown real number. Show that for any real number $\alpha \in (0,1)$,

$$\alpha(X_{(n)}-1)+(1-\alpha)(X_{(1)}+1)$$

is a maximum likelihood estimator for the unknown θ, where $X_{(1)}$ and $X_{(n)}$ are the smallest and largest sample observations respectively.

(b) Let $X_1, \dots, X_n$ be iid $N(\mu, 1)$ where μ is only known to belong to the set of all integers. Find a maximum likelihood estimator for μ based on $X_1, \dots, X_n$.

S5. Suppose $X_1, \dots, X_n$ are independent and identically distributed random variables following $N(\theta, 1)$, $\theta \in \boldsymbol{R}$. Let $\varphi(\theta) = P_\theta(X_1 > u_0)$, where u_0 is a known real number. Show that the uniformly minimum variance unbiased estimate (UMVUE) of $\varphi(\theta)$ is given by

$$T(X_1, \dots, X_n) = 1 - \Phi\left[\sqrt{\frac{n}{n-1}}(u_0 - \overline{X})\right],$$

where $\varphi(\cdot)$ is the distribution function of the standard normal distribution.

S6. Consider a randomized block design with two blocks and two treatments A and B. The following table gives the yields:

	Treatment A	Treatment B
Block 1	a	b
Block 2	c	d

(a) How many orthogonal contrasts are possible with a, b, c and d? Write down all of them.

(b) Identify the contrasts representing block effects, treatment effects and error.

(c) Show that their sum of squares equals the total sum of squares.

S7. Let X be a discrete random variable having the probability mass function

$$p(x) = \Lambda^x(1-\Lambda)^{1-x},\ x = 0, 1,$$

where Λ takes values ≥ 0.5 only. Find the most powerful test, based on 2 observations, for testing $H_0 : \Lambda = \frac{1}{2}$ against $H_1 : \Lambda = \frac{2}{3}$, with level of significance 0.05.

S8. Let $X=(X_1,\ldots, X_n)$ be a random sample from the exponential distribution $E(\theta, \sigma)$ having unknown location parameter θ and unknown scale parameter σ. Consider the problem of testing H_0: $\theta = \theta_0$ against H_1: $\theta \neq \theta_0$.

(a) Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistics associated with **X**. Let

$$T = \frac{X_{(1)} - \theta_0}{\sum_{i=1}^{n}(X_i - X_{(1)})}.$$

Find the null distribution of T in terms of an F-distribution, with degrees of freedom to be obtained by you.

(b) Fix $0 < \alpha < 1$, Find C_1, C_2 with $0 < C_1 < C_2$ such that the test with rejection region "$T \leq C_1$ or C_2" has size α.

(c) Show that for any alternative (θ_1, σ) with $\theta_1 < \theta_0$, the power of the level-α test in (b), denoted by $\beta(\theta_1, \sigma)$, is given by

$$\beta(\theta_1, \sigma) = 1 - (1-\alpha)\exp\{-n(\theta_0 - \theta_1)/\sigma\}$$

S9. Let $X=(X_1,\ldots, X_n)$ be a sample from the uniform distribution on $(0, \theta)$. Show the following:

(a) For testing H_0: $\theta \leq \theta_0$ against H_1: $\theta \geq \theta_0$, any test is UMP at level α for which $E_{\theta_0}(\phi(X)) = \alpha$, $E_{\theta_0}(\phi(X)) \leq \alpha$ for $\theta \leq \theta_0$, and $\phi(x) = 1$ when $\max(x_1, \ldots, x_n) > \theta_0$.

(b) For testing H_0: $\theta = \theta_0$ against H_1: $\theta \neq \theta_0$, a unique UMP test exits, and is given by ϕ(x) = 1 when max $(x_1,....,x_n) > \theta_0$ or max $(x_1,....,x_n) \leq \theta_0 \alpha^{1/n}$ and ϕ(x) = 0 otherwise.

S10. Consider a population with 3 units, labeled 1, 2 and 3. Let the values of a random variable of interest (y) for these units be y_1, y_2 and y_3 respectively. A simple random sample without replacement (SRSWOR) of size 2 is drawn from this population. Consider the two estimators:

(a) $\bar{y}$, i.e., the usual sample mean and

(b) $$\hat{\hat{Y}} = \begin{cases} \dfrac{y_1}{2} + \dfrac{y_2}{2} & \text{if the sample consists of units 1 and 2} \\ \dfrac{y_1}{2} + \dfrac{2y_3}{3} & \text{if the sample consists of units 1 and 3} \\ \dfrac{y_2}{2} + \dfrac{y_3}{3} & \text{if the sample consists of units 2 and 3} \end{cases}$$

(i) Show that both estimators are unbiased for the population mean.

(ii) Show that $\text{Var}(\hat{\hat{Y}}) < \text{Var}(\bar{y})$ if $y_3(3y_2 - 3y_1 - y_3) > 0$.

(Hint: Suppose $\bar{x}$ is the sample mean for a simple random sample without replacement of size n from a population of size N with population unit values $x_1, \dots, x_N$. Then

$$\text{Var}(\bar{x}) = \frac{N-n}{Nn} \frac{\sum_{i=1}^{N}(x_i - \bar{X})^2}{N-1},$$

where $\bar{X} = \dfrac{\sum_{i=1}^{N} x_i}{N}$.)

S11. Suppose $X_1, \dots, X_n$ are i.i.d. exponential variables with locations parameter $\theta > 0$ and scale parameter 1. Let $X_{(1)} = \min\{X_1, ..., X_n\}$.

(a) Show that the distribution function of $T = X_{(1)}$, denoted by $F_\theta(t)$, is a decreasing function of θ.

(b) Given α $(0 < \alpha < 1)$, use (a) to obtain a $(1-\alpha)$ confidence interval for θ.

S12. Let $X_1, X_2, \ldots, X_n$ ($X_i = (x_{i1}, x_{i2}, \ldots, x_{ip}),\ i=1, 2, \ldots, n$) be n random samples from a p-variate normal population with mean vector $\underset{\sim}{\mu}$ and covariance matrix I.

Further, let S = $((s_{jk}))$ denote the sample sums of squares and products matrix, namely

$$s_{jk} = \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k), 1 \le j, k \le p, \text{ where}$$

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}, 1 \le j \le p.$$

Obtain the distribution of $\ell' S \ell$ where $\ell \in \Re^k, \ell \ne 0$.

S13. Suppose $\mathbf{X} = (X_1, X_2, X_3)^T \sim N_3(\underset{\sim}{\mu}, \Sigma)$, where

$$\tilde{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

Show that $E(X_1 X_2 X_3) = \mu_1 \mu_2 \mu_3 + \mu_1 \sigma_{23} + \mu_2 \sigma_{31} + \mu_3 \sigma_{12}$.

(b) Suppose $\mathbf{X} = (X_1, X_2, X_3, X_4)^T \sim N_4(\underset{\sim}{0}, \Sigma)$, where $\Sigma = ((\sigma_{ij}))$.

Show that $E(X_1 X_2 X_3 X_4) = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}$.

S14. An experimenter wants to study three factors, each at two levels, for their individual effects and interaction effects, if any. If the experimental units are heterogeneous with respect to two factors of classification, suggest a suitable experimental design for the study. Give the analysis of variance (ANOVA) for the suggested design, indicating clearly how the various sums of squares are to be computed.

Physics

P1. (a) In a photoelectric emission experiment, a metal surface was successively exposed to monochromatic lights of wavelength λ_1, λ_2 and λ_3. In each case, the maximum velocity of the emitted photo electrons was measured and found to be α, β and γ, respectively. λ_3 was *10%* higher in value than λ_1, whereas λ_2 was *10%* lower in value than λ_1. If $\beta : \gamma = 4 : 3$, then show that

$$\alpha : \beta = 9\sqrt{3} : 8\sqrt{5}.$$

(b) The quantum number of two electrons in a two-valence electron are;

$$n_1 = 6;\ l_1 = 3;\ s_1 = 1/2$$
$$n_2 = 5;\ l_2 = 1;\ s_2 = 1/2$$

(i) Assuming L-S coupling, find the possible values of L and hence J.
(ii) Assuming J-J coupling, find the possible values of J.

P2. (a) Consider a material that has two solid phases, a metallic phase and an insulator phase. The phase transition takes place at the temperature T_0 which is well below the Debye temperature for either phase. The high temperature phase is metastable all the way down to $T = 0$ and the speed of sound, c_s, is the same for each phase. The contribution to the heat capacity coming from the free electrons to the metal is

$$C_e = \rho_e V_\gamma T \ , \qquad \gamma = 3\pi^2 \frac{k}{4T_F}$$

where ρ_e is the number density of the free electrons, T_F is the Fermi temperature, K is the Boltzmann constant, and V is the volume. Calculate the latent heat per unit volume required to go from the low temperature phase to the high temperature phase at $T = T_0$. Which phase is the high temperature phase?

(c) A crystal at temperature T is made up of N noninteracting atoms, where each atom can be in one of two states, the energies of these states being B.
(i) Find the partition function of this system.
(ii) Find the energy of this system.
(iii) Find the entropy of this system and then give the expression in the limit of very low and very high temperatures.

P3. (a) A particle of mass m moves under a force directed towards a fixed point and this force depends on the distance from the fixed point. Show that

(i) the particle will be constrained to move in a plane, and

(ii) the areal velocity of the particle is constant.

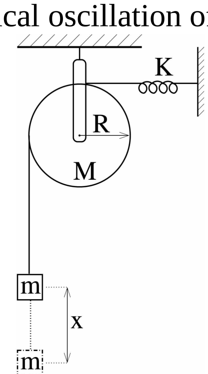
(b) If the force **F** varies as the inverse of the square of the distance, show that

$$\nabla \times \mathbf{F} = 0.$$

Discuss its implications.

(c) Assuming the trajectory of planets to be circular, deduce the force law from Kepler's third law.

P4. (a) A mass m is attached to a massless spring of spring constant K via a frictionless pulley of radius R and mass M as shown in following figure. The mass m is pulled down through a small distance x and released, so that it is set into simple harmonic motion. Find the frequency of the vertical oscillation of the mass m.



(b) The Hamiltonian of a mechanical system having two degrees of freedom is:

$$H(x, y; p_x, p_y) = \frac{1}{2m}\,(p_x^2 + p_y^2) + \frac{1}{2}\,m\,\omega^2\,(x^2 + y^2),$$

where m and ω are constants; x, y are the generalized co-ordinates for which p_x, p_y are the respective conjugate momenta. Show that the expressions $(x\,p_y - y\,p_x)^n$, $n=1,2,3,\ldots$ are constants of motion for this system.

P5. (a) Consider a hollow sphere of radius r having surface density of mass equal to ¾. Consider any point inside the sphere which is at a distance a from the origin. Find the gravitational force and potential at that point due to the mass of the sphere.

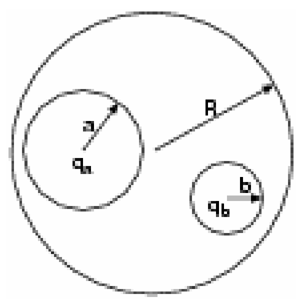
(b) In a Millikan's oil drop experimental setup, two small negatively charged spherical oil droplets having radii $3r$ and $5r$ were allowed to fall freely in the closed chamber filled with air. The downward terminal velocities attained by them were v_1 and v_2 respectively. Subsequently, under the action of a strong electric field, the droplets attained upward terminal velocities $v_1/6$ and $v_2/20$ respectively. Neglecting the bouyant force of air and assuming the charges to be uniformly distributed over the surface of the droplets, compare their surface charge densities.

P6. (a) A dielectric sphere of radius R and permittivity ε carries a volume charge density $\rho(r) = kr$ (where k is a constant). Deduce an expression for the energy of the configuration.

(b) Two spherical cavities of radii a and b are hollowed out from the interior of a neutral conducting sphere of radius R. Two point charges of magnitude q_a and q_b are now placed at the centres of the two cavities as shown in the figure.

(ii)Calculate the surface charge densities on the surfaces of the two spherical cavities and the sphere.

(iii)What are the magnitudes of the forces on q_a and q_b?



P7. A train passes a platform with velocity v. Two clocks are placed on the edge of the platform separated by a distance L and synchronized relative to platform inertial system. Clock 1 reads time t_1 when it coincides with the front of the train and clock 2 reads time t_2 when it coincides with the rear of the train. Answer the following questions relative to an observer on the train.

(a) What is the length of the train?

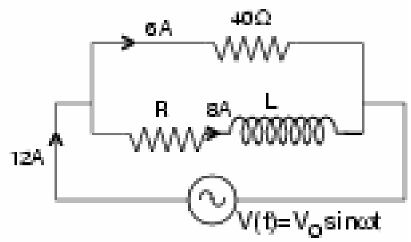
(b) What is the reading of clock 2 when the clock1 coincides with the front of the train?

(c) What is the time interval between the two end events?

P8. (a) A perfect gas expands in a manner such that its elasticity is always equal to the sum of the isothermal and adiabatic elasticity. Find its specific heat under this condition in terms of the specific heats at constant pressure and constant volume.

(b) Two bodies A and B of equal and constant thermal capacity c were initially at absolute temperatures T_A and T_B ($T_A > T_B$), respectively. A reversible heat engine acting between them does some amount of external work W so that A and B finally attain the same temperature T. Find expressions for T and W in terms of c, T_A, and T_B.

P9. In the circuit shown below, the peak current flowing through the different branches are indicated. Derive the value of the total power delivered by the source.



P10. Two heavy bodies A and B, each having charge $-Q$, are kept rigidly fixed at a distance $2a$ apart. A small particle C of mass m and charge $+q$ ($<< Q$), is placed at the midpoint of the straight line joining the centers of A and B. C is now displaced slightly along a direction perpendicular to the line joining A and B, and then released. Find the period of the resultant oscillatory motion of C, assuming its displacement $y << a$.

If instead, C is slightly displaced towards A, then find the instantaneous velocity of C, when the distance between A and C is $\frac{a}{2}$.

P11. An elementary particle called Σ-, at rest in laboratory frame, decays spontaneously into two other particles according to $\Sigma- \rightarrow \pi^- + n$. The masses of Σ-, π^- and n are M_1, m_1, and m_2 respectively.

(a)How much kinetic energy is generated in the decay process?

(b)What are the ratios of kinetic energies and momenta of π^- and n?

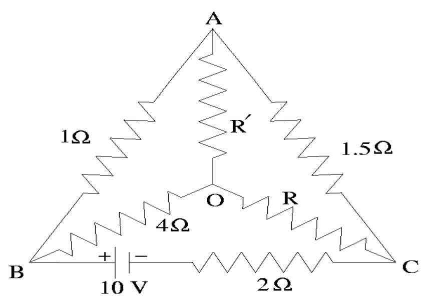
P12. Consider the following truth table where *A, B* and *C* are Boolean inputs and *T* is the Boolean output.

A	B	C	T
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Express *T* in a product-of-sum form and hence, show how *T* can be implemented using NOR gates only.

P13. (a) A series R-L-C circuit is excited from a constant-peak variable frequency voltage source of the form $V = V_0 \sin\omega t$, where V_0 is constant. The current in the circuit becomes maximum at a frequency of $\omega_0 = 600$ rad/sec. and falls to half of the maximum value at $\omega = 400$ rad/sec. If the resistance in the circuit is 3Ω, find L and C.

(b) Find the value of *R* and the current flowing through *R* shown in the figure when the current is zero through *R′*.



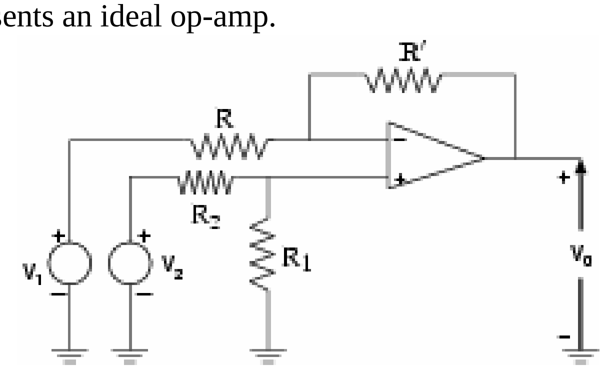
P14. A gas obeys the equation of state $P = \frac{\tau}{V} + \frac{B(\tau)}{V^2}$ where $B(\tau)$ is a function of temperature τ only. The gas is initially at temperature τ and volume V_0 and is expanded isothermally and reversibly to volume $V_1 = 2V_0$.

(a) Find the work done in the expansion.

(b) Find the heat absorbed in the expansion.

(*Hint: Use the relation* $\left(\frac{\partial S}{\partial V}\right)_\tau = \left(\frac{\partial P}{\partial \tau}\right)_V$ *where the symbols have their usual meaning.*)

P15. Consider the following circuit where the triangular symbol represents an ideal op-amp.



(a) Calculate the output voltage v_0 for the (i) common-mode operation and (ii) difference mode operation.

(b) Also calculate the value of the common-mode rejection ratio for $R'/R = R_1/R_2$.

P16. (a) A particle of mass m is moving in a plane under the action of an attractive force proportional to $1/r^2$, r being the radial distance of the particle from the fixed point. Write the Lagrangian of the system and using the Lagrangian show that the areal velocity of the particle is conserved (Kepler's second law).

(b) A particle of mass m and charge q is moving in an electro-magnetic field with velocity v. Write the Lagrangian of the system and hence find the expression for the generalized momentum.

Computer Science

C1. (a) A grammar is said to be left recursive if it has a non-terminal A such that there is a derivation $A \Rightarrow^{+} A\alpha$ for some sequence of symbols α. Is the following grammar left-recursive? If so, write an equivalent grammar that is not left-recursive.

$$A \rightarrow Bb \qquad A \rightarrow a$$
$$B \rightarrow Cc \qquad B \rightarrow b$$
$$C \rightarrow Aa \qquad C \rightarrow c$$

(b) An example of a function definition in C language is given below:

```
char fun (int a, float b, int c)
{ /* body */ … }
```

Assuming that the only types allowed are `char`, `int`, `float` (no arrays, no pointers, etc.), write a grammar for function headers, i.e., the portion `char fun(int a, …)` in the above example.

(c) Consider the floating point number representation in the C programming language.
Give a regular expression for it using the following convention: l denotes a letter, d denotes a digit, S denotes sign and p denotes point.
State any assumption that you may need to make.

C2. The following functional dependencies are defined on the relation $\Re(A, B, C, D, E, F)$:

$$\{\ A \to B,\ AB \to C,\ BC \to D,\ CD \to E,\ E \to A\ \}$$

(a) Find the candidate keys for $\Re$.

(b) Is $\Re$ normalized? If not, create a set of normalized relations by decomposing $\Re$ using only the given set of functional dependencies.

(c) If a new attribute F is added to $\Re$ to create a new relation $\Re'(A, B, C, D, E, F)$ without any addition to the set of functional dependencies, what would be the new set of candidate keys for $\Re'$?

(d) What is the new set of normalized relations that can be derived by decomposing $\Re'$ for the same set of functional dependencies?

(e) If a new dependency is declared as follows:

For each value of A, attribute F can have two values,

what would be the new set of normalized relations that can be created by decomposing $\Re'$?

C3.(a) A relation *R(<u>A</u>, B, C, D)* has to be accessed under the query $\sigma_{B=10}(R)$. Out of the following possible file structures, which one should be chosen and why?

i) *R* is a heap file.

ii) *R* has a clustered hash index on *B*.

iii) *R* has an unclustered B^+ tree index on *(A, B)*.

(b) If the query is modified as $\pi_{A,B}(\sigma_{B=10}(R))$, which one of the three possible file structures given above should be chosen in this case and why?

(c) Let the relation have 5000 tuples with 10 tuples/page. In case of a hashed file, each bucket needs 10 pages. In case of B^+ tree, the index structure itself needs 2 pages. If it takes 25 msecs. to read or write a disk page, what would be the disk access time for answering the above queries?

(d) Relation R(*A,B,C*) supports the following functional dependencies:

$$A \to B,\ B \to C \text{ and } C \to A.$$

(i) Identify the key attributes.

(ii) Explain whether R is in BCNF.

(iii) If R is not in BCNF, decompose to create a set of normalized relations satisfying BCNF.

(iv) If R does not support the functional dependencies $B \rightarrow C$, but the other two are maintained, would R be in BCNF? If not, decompose R to normalized relations satisfying BCNF.

C4. Let A and B be two arrays, each of size n. A and B contain numbers in sorted order. Give an $O(\log n)$ algorithm to find the median of the combined set of $2n$ numbers.

C5. (a) Consider a pipelined processor with m stages. The processing time at every stage is the same. What is the speed-up achieved by the pipelining?

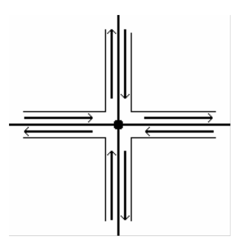
(b) In a certain computer system with cache memory, 750 ns (nanosec) is the access time for main memory for a cache miss and 50 ns is the access time for a cache hit. Find the percentage decrease in the effective access time if the hit ratio is increased from 80% to 90%.

C6. (a) A disk has 500 bytes/sector, 100 sectors/track, 20 heads and 1000 cylinders. The speed of rotation of the disk is 6000 rpm. The average seek time is 10 millisecs. A file of size 50 MB is written from the beginning of a cylinder and a new cylinder will be allocated only after the first cylinder is totally occupied.

i) Find the maximum transfer rate.

ii) How much time will be required to transfer the file of 50 MB written on the disk? Ignore the rotational delay but not the seek time.

(b) Consider a 4-way traffic crossing as shown in the figure.



Suppose that we model the crossing as follows:

- each vehicle is modeled by a process,
- the crossing is modeled as a shared data structure. Assume that the vehicles can only move straight through the intersection (no left or right turns). Using read-write locks (or any standard synchronization primitive), you have to device a synchronization scheme for the processes. Your scheme should satisfy the following criteria:

i) prevent collisions,
ii) prevent deadlock, and
iii) maximize concurrency but prevent indefinite waiting (starvation).

Write down the algorithm that each vehicle must follow in order to pass through the crossing. Justify that your algorithm satisfies the given criteria.

C7. (a) A computer on a 6 Mbps network is regulated by a token bucket. The bucket is filled at a rate of 2 Mbps. It is initially filled to capacity with 8 Megabits. How long can the computer transmit at the full 6 Mbps?

(b) Sketch the Manchester encoding for the bit stream 0001110101.

(c) If delays are recorded in 8-bit numbers in a 50-router network, and delay vectors are exchanged twice a second, how much bandwidth per (full-duplex) line is consumed by the distributed routing algorithm? Assume that each router has 3 lines to other routers.

(d) Consider three IP networks X, Y, and Z. Host H_X in the network X sends messages, each containing 180 bytes of application data, to a host H_Z in network Z. The TCP layer prefixes a 20 byte header to the message. This passes through an intermediate network Y. The maximum packet size, including 20 byte IP header, in each network is X: 1000 bytes, Y: 100 bytes, and Z: 1000 bytes. The networks X and Y are connected through a 1 Mbps link, while Y and Z are connected by a 512 Kbps link.

(i) Assuming that the packets are correctly delivered, how many bytes, including headers, are delivered to the IP layer at the destination for one application message? Consider only data packets.

(ii) What is the rate at which application data is transferred to host H_Z? Ignore errors, acknowledgements, and other overheads.

C8. Consider a binary operation *shuffle* on two strings, that is just like shuffling a deck of cards. For example, the operation *shuffle* on strings *ab* and *cd*, denoted by *ab* || *cd*, gives the set of strings {*abcd*, *acbd*, *acdb*, *cabd*, *cadb*, *cdab*}.

(a) Define formally by induction the *shuffle* operation on any two strings $x, y \in \Sigma^*$.

(b) Let the *shuffle* of two languages *A* and *B*, denoted by *A* || *B* be the set of all strings obtained by shuffling a string $x \in A$ with a string $y \in B$. Show that if *A* and *B* are regular, then so is *A* || *B*.

C9. (a) Give a method of encoding the microinstructions (given in the table below) so that the minimum number of control bits are used and maximum parallelism among the microinstructions is achieved.

Microinstructions	Control signals
I_1	$C_1, C_2, C_3, C_4, C_5, C_6,$
I_2	$C_1, C_3, C_4 C_6,$
I_3	$C_2, C_5, C_6,$
I_4	$C_4, C_5, C_8,$
I_5	$C_7, C_8,$
I_6	$C_1, C_8, C_9,$
I_7	$C_3, C_4, C_8,$
I_8	$C_1, C_2, C_9,$

(b) A certain four-input gate *G* realizes the switching function

$$G(a, b, c, d) = abc + bcd.$$

Assuming that the input variables are available in both complemented and uncomplemented forms:

(i) Show a realization of the function

$$f(u, v, w, x) = \Sigma(0, 1, 6, 9, 10, 11, 14, 15)$$

with only three G gates and one OR gate.
(ii) Can all switching functions be realized with $\{G, OR\}$ logic set?

C10. Consider a set of n temperature readings stored in an array T. Assume that a temperature is represented by an integer. Design an $O(n + k \log n)$ algorithm for finding the k coldest temperatures.

C11. Assume the following characteristics of instruction execution in a given computer:

- ALU/register transfer operations need 1 clock cycle each,
- each of the load/store instructions needs 3 clock cycles, and
- branch instructions need 2 clock cycles each.

(a) Consider a program which consists of 40% ALU/register transfer instructions, 30% load/store instructions, and 30% branch instructions. If the total number of instructions in this program is 10 billion and the clock frequency is 1 GHz, then compute the average number of cycles per instruction (CPI), total execution time for this program, and the corresponding MIPS rate.
(b) If we now use an optimizing compiler which reduces the total number of ALU/register transfer instructions by a factor of 2, keeping the number of other instruction types unchanged, then compute the average CPI, total time of execution and the corresponding MIPS rate for this modified program.

C12. Consider a computer system with 1 GB main memory and 1 MB cache memory organized in blocks of 64 bytes.
(a) What is the minimum number of bits needed for addressing a memory location?
(b) How many bits are needed for the tag field and the index field if the cache memory is organized in the following ways: (i) direct-mapped, (ii) fully associative, and (iii) 2-way set-associative?
(c) Suppose the memory location to be accessed is 000D0237 (in hex). What cache block will be accessed for this memory location in the direct-mapped organization and what will be the value of the tag field? If instead, the cache memory were organized in a fully associative manner, what will be the corresponding value of the tag field?

(d) Express the following numbers in IEEE 754-1985 single precision floating-point format:

(i) -0 (ii) 2.5×2^{-130} (iii) 230 (iv) 0.875 (v) $(-3)^{1/8}$.

C13. A tape *S* contains *n* records, each representing a vote in an election. Each candidate for the election has a unique *id*. A vote for a candidate is recorded as his/her *id*.

(i) Write an *O(n)* time algorithm to find the candidate who wins the election. Comment on the main memory space required by your algorithm.

(ii) If the number of candidates *k* is known *a priori*, can you improve your algorithm to reduce the time and/or space complexity?

(iii) If the number of candidates *k* is unknown, modify your algorithm so that it uses only *O(k)* space. What is the time complexity of your modified algorithm?

C14. (a) The *order* of a regular language *L* is the smallest integer *k* for which $L^k = L^{k+1}$, if there exists such a *k*, and ∞ otherwise.

(i) What is the order of the regular language $a + (aa)(aaa)^*$?

(ii) Show that the order of *L* is finite if and only if there is an integer *k* such that $L^k = L^*$, and that in this case the order of *L* is the smallest *k* such that $L^k = L^*$.

(b) Solve for *T(n)* given by the following recurrence relations:

$T(1) = 1;$

$T(n) = 2T(n/2) + n \log n$, where *n* is a power of 2.

(c) An A.P. is $\{p + qn | n = 0, 1, \ldots\}$ for some *p, q* □ *IN*. Show that if *L* □ $\{a\}^*$ and $\{n | a^n$ □ $L\}$ is an A.P., then *L* is regular.

C15. (a) You are given an unordered sequence of *n* integers with many duplications, such that the number of distinct integers in the sequence is *O(log n)*. Design a sorting algorithm and its necessary data structure(s), which can sort the sequence using at most *O(n log(log n))* time. (You have to justify the time complexity of your proposed algorithm.)

(b) Let *A* be a real-valued matrix of order *n x n* already stored in memory. Its *(i, j)*-th element is denoted by *a[i, j]*. The elements of the matrix *A* satisfy the following property:

Let the largest element in row *i* occur in column l_i. Now, for any two rows i_1, i_2, if $i_1 < i_2$, then $l_{i1} \leq l_{i2}$.

2	6	4	5	3
5	3	7	2	4
4	2	10	7	8
6	4	5	9	7
3	7	6	8	12

(a)

Row *I*	*l*(*i*)
1	2
2	3
3	3
4	4
5	5

(b)

Figure shows an example of (a) matrix A, and (b) the corresponding values of l_i for each row *i*.

Write an algorithm for identifying the largest valued element in matrix *A* which performs at most $O(n\log_2 n)$ comparisons.

C16. You are given the following file abc.h:

```
#include <stdio.h>
#define SQR(x) (x*x)
#define ADD1(x) (x=x+1)
#define BeginProgram int main(int ac,char *av[]){
#define EndProgram return 1; }
```

For each of the following code fragments, what will be the output?

(i)
```
#include "abc.h"
main()
{ int y = 4; printf("%d\n", SQR(y+1)); }
```

(ii)
```
#include "abc.h"
BeginProgram
int y=3; printf("%d\n", SQR(ADD1(y)));
EndProgram
```

Engineering and Technology

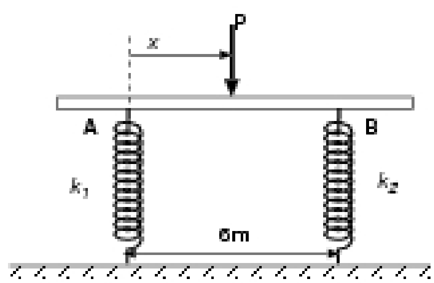
E1. A bullet of mass M is fired with a velocity of 40 m/s at an angle θ with the horizontal plane. At P, the highest point of its trajectory, the bullet collides with a bob of mass 3M suspended freely by a mass-less string of length $\frac{3}{10}$ m. After the collision, the bullet gets stuck inside the bob and the string deflects with the total mass through an angle of 120^o keeping the string taut. Find

(i) the angle θ, and

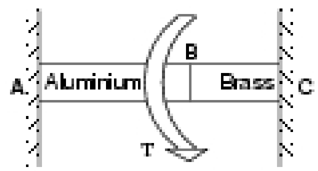
(ii) the height of P from the horizontal plane.

Assume, g = 10 m/s^2, and friction in air is negligible.

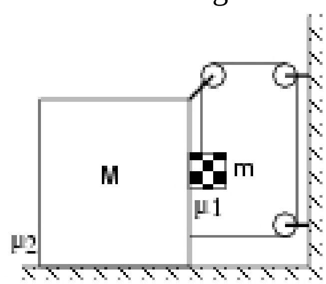
E2. (a) A rigid horizontal bar of negligible weight is supported by two springs as shown in the figure below. Determine the distance *x* in order that the bar remains horizontal after a load *P* is applied.



(b) A composite shaft of Aluminium and Brass is rigidly supported at the ends *A* and *C*, as shown in the figure below. The shaft is subjected to a shearing stress by the application of a torque *T*. Calculate the ratio of lengths *AB* : *BC* if each part of the shaft is stressed to its maximum limit (beyond which the composite shaft will break). Assume the maximum shear stress of Brass and Aluminium to be 560 kg/cm^2 and 420 kg/cm^2 respectively. Also assume that the modulus of rigidity of Brass is twice that of Aluminium.

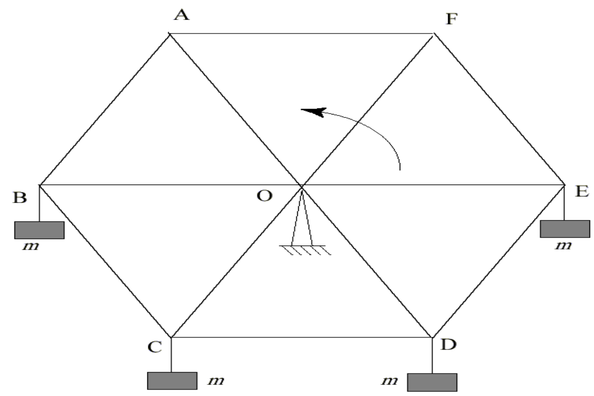


E3. Find the acceleration of the block of mass *M* in the situation shown below. The coefficient of friction between the blocks is μ_1 and that between the bigger block and the ground is μ_2.



E4. A flywheel of mass 100 kg and radius of gyration 20 cm is mounted on a light horizontal axle of radius 2 cm, and is free to rotate on bearings whose friction may be neglected. A light string wound on the axle carries at its free end a mass of 5 kg. The system is released from rest with the 5 kg mass hanging freely. If the string slips off the axle after the weight has descended 2 m, prove that a couple of moment $10/\pi^2$ kg.wt.cm. must be applied in order to bring the flywheel to rest in 5 revolutions.

E5. The truss shown in the figure rotates around the pivot O in a vertical plane at a constant angular speed ω. Four equal masses (*m*) hang from the points B, C, D and E. The members of the truss are rigid, weightless and of equal length. Find a condition on the angular speed ω so that there is compression in the member OE.

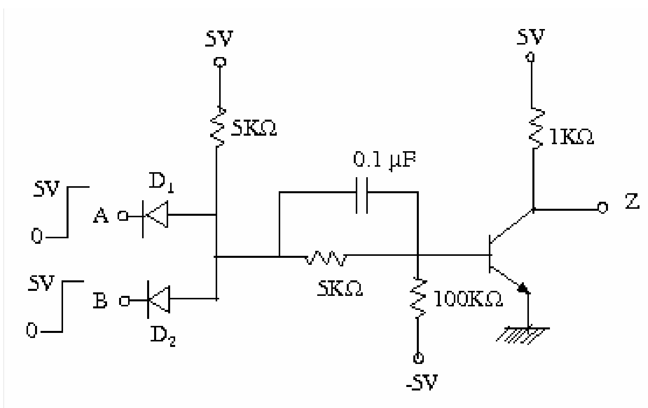


E6. If the inputs A and B to the circuit shown below can be either 0 volt or 5 volts,

(i) what would be the corresponding voltages at output Z, and

(ii) what operation is being performed by this circuit ?

Assume that the transistor and the diodes are ideal and base to emitter saturation voltage = 0.5 volts.



E7. Two bulbs of 500 cc capacity are connected by a tube of length 20 cm and internal radius 0.15 cm. The whole system is filled with oxygen, the initial pressures in the bulbs before connection being 10 cm and 15 cm of Hg, respectively. Calculate the time taken for the pressures to become 12 cm and 13 cm of Hg, respectively. Assume that the coefficient of viscosity η of oxygen is 0.000199 cgs unit.

E8. (a) Ice in a cold storage melts at a rate of $\frac{300\times3.6}{80\times4.2}$ *kg/hour* when the external temperature is 27^oC. Find the minimum power output of the refrigerator motor, which just prevents the ice from melting. (Latent heat of fusion of ice = 80 *cal/gm*.)

(b) A vertical hollow cylinder contains an ideal gas with a 5 kg piston placed over it. The cross-section of the cylinder is $5\times10^{-3}\ m^2$. The gas is heated from 300 K to 350 K and the piston rises by 0.1 *m*. The piston is now clamped in this position and the gas is cooled back to 300 K. Find the difference between the heat energy added during heating and that released during cooling.
(1 atmospheric pressure= $10^5 Nm^{-2}$ and $g=10ms^{-2}$.)

E9. (a) A system receives 10 Kcal of heat from a reservoir to do 15 Kcal of work. How much work must the system do to reach the initial state by an adiabatic process?

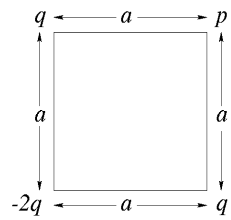
(b) A certain volume of Helium at 15˚C is suddenly expanded to 8 times its volume. Calculate the change in temperature (assume that the ratio of specific heats is 5/3).

E10. A spherical charge distribution has a volume density ρ, which is a function of r, the radial distance from the center of the sphere, as given below.

$$\rho = \begin{cases} A/r,\ A \text{ is constant } for\ 0 \le r \le R \\ 0, \quad \text{for}\ r > R \end{cases}$$

Determine the electric field as a function of r, for $r \ge R$. Also deduce the expression for the electrostatic potential energy $U(r)$, given that $U(\infty) = 0$ in the region $r \ge R$.

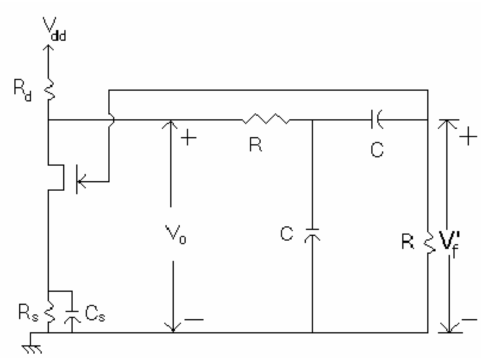
E11. Consider the distribution of charges as shown in the figure below. Determine the potential and field at the point *p*.



E12. A proton of velocity 10^7 m/s is projected at right angles to a uniform magnetic induction field of 0.1 w/m^2. How much is the path of the particle deflected from a straight line after it has traversed a distance of 1 cm? How long does it take for the proton to traverse a 90^0 arc?

E13. (a) State the two necessary conditions under which a feedback amplifier circuit becomes an oscillator.

(b) A two-stage FET phase shift oscillator is shown in the diagram below.



(i) Derive an expression for the feedback factor β.
(ii) Find the frequency of oscillation.
(iii) Establish that the gain A must exceed 3.

E14. A circular disc of radius 10cm is rotated about its own axis in a uniform magnetic field of 100 weber/m^2, the magnetic field being perpendicular to the plane of the disc. Will there be any voltage developed across the disc? If so, then find the magnitude of this voltage when the speed of rotation of the disc is 1200 rpm.

E15. A 3-phase, 50-Hz, 500-volt, 6-pole induction motor gives an output of 50 HP at 900 rpm. The frictional and windage losses total 4 HP and the stator losses amount to 5 HP. Determine the slip, rotor copper loss, and efficiency for this load.

E16. A d.c. shunt motor running at a speed of 500rpm draws 44KW power with a line voltage of 220V from a d.c. shunt generator. The field resistance and the armature resistance of both the machines are 55 Ω and 0.025 Ω respectively. However, the voltage drop per brush is 1.05V in the motor, and that in the generator is 0.95V. Calculate

(a) the speed of the generator in rpm, and

(b) the efficiency of the overall system ignoring losses other than the copper-loss and the loss at the brushes.

E17. An alternator on open-circuit generates 360 V at 60 Hz when the field current is 3.6 A. Neglecting saturation, determine the open-circuit e.m.f. when the frequency is 40 Hz and the field-current is 24A.

E18. A single phase two-winding *20 KVA* transformer has 5000 primary and 500 secondary turns. It is converted to an autotransformer employing additive polarity mechanism. Suppose the transformer always operates with an input voltage of 2000 *V*.

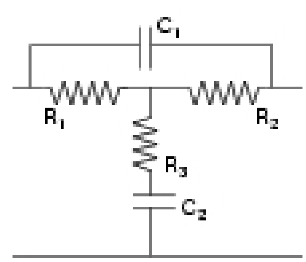
(i) Calculate the percentage increase in KVA capacity.

(ii) Calculate the common current in the autotransformer.

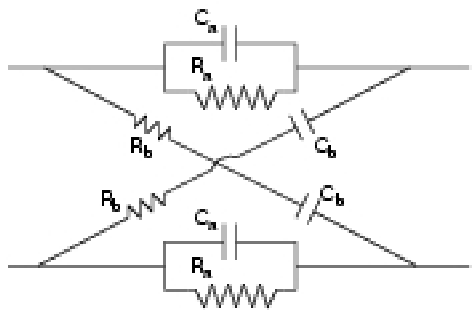
(iii) At full load of 0.9 power factor, if the efficiency of the two-winding transformer be 90%, what will be the efficiency of the autotransformer at the same load?

E19. The hybrid parameters of a *p-n-p* junction transistor used as an amplifier in the common-emitter configuration are: h_{ie} = 800Ω, h_{fe} = 46, h_{oe} = 8 x 10^{-5} mho, h_{re} = 55.4 x 10^{-4}. If the load resistance is 5 *k*Ω and the effective source resistance is 500 Ω, calculate the voltage and current gains and the output resistance.

E20. (a) Derive the equivalent lattice network corresponding to the bridged *T* network shown in the figure.



(b) Find the open-circuit transfer impedance of the lattice shown in the figure below and determine the condition for having no zeros in the right-half plane, i.e., for positive frequencies.

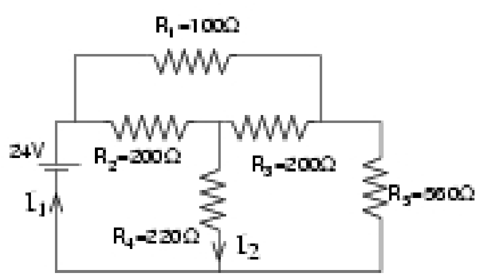


E21. A logic circuit operating on Binary Coded Decimal (BCD) digits has four inputs X_1, X_2, X_3, and X_4, where $X_1X_2X_3X_4$ represents a BCD digit. The circuit has two output lines Z_1 and Z_2. Output Z_1 is 1 only when the decimal digit corresponding to the inputs X_1, X_2, X_3, X_4 is 0 or a power of 2. Output Z_2 is 1 only when the decimal digit corresponding to the inputs is 1 or a power of 3. Find a minimum cost realization of the above circuit using NAND gates.

E22. (a) Using the minimum number of flip-flops, design a special purpose counter to provide the following sequence:

0110, 1100, 0011, 1001 (returning from 1001 to 0110)

(b) Find the currents I_1 and I_2 in the following circuit.



E23. Write a C program to generate a sequence of positive integers between 1 and N, such that each of them has only 2 or/and 3 as prime factors. For example, the first seven elements of the sequence are: 2, 3, 4, 6, 8, 9, 12. Justify the steps of your algorithm.

E24. Design a circuit using the module, as shown in the figure below, to compute a solution of the following set of equations:

$$3x + 6y - 10 = 0$$
$$2x - y - 8 = 0$$

A module consists of an ideal OP-AMP and 3 resistors, and you may use multiple copies of such a module. Voltage inverters and sources may be used, if required.