JRF IN MATHEMATICS 2001-2002

There will be two tests, RMI and RMII, of 2 hours duration each in the forenoon and in the afternoon. Topics to be covered in these tests along with an outline of the syllabus and sample questions are given below:

1. Topics for RMI (Forenoon examination) : Real Analysis, Complex Analysis, Algebra, Linear Algebra and General Topology.

2. Topics for RMII (Afternoon examination) : Lebesgue measure and integration, Functional Analysis, Elementary Number theory and Combinatorics, and Ordinary differential equations.

Candidates will be judged based on their performance in **both** the tests.

OUTLINE OF THE SYLLABUS

1. **General Topology :** Topological spaces, Continuous functions and homeomorphisms, Connectedness, Compactness, Separation Axioms. Urysohn Lemma and Tietze extension theorem. Product spaces. Complete metric spaces. Uniform continuity. Baire category theorem.
2. **Functional Analysis :** Normed linear spaces, Banach spaces, Hilbert spaces, Compact operators. Knowledge of some standard examples like $C[0,1]$, $L^p[0,1]$. Continuous linear maps (linear operators). Hahn-Banach Theorem, Open mapping theorem, Closed graph theorem and the uniform boundedness principle.
3. **Real analysis :** Continuity and differentiability of real valued functions of one and several real variables and applications; sequences and series of functions, uniform convergence, Riemann integration.
4. **Linear algebra :** Vector spaces, basis and dimension; linear transformations, matrices, characteristic roots and characteristic vectors, systems of linear equations, inner product spaces, projections, diagonalization of symmetric and Hermitian matrices, quadratic forms.
5. **Elementary number theory :** Divisibility, congruences, standard arithmetic functions, permutations and combinations.
6. **Lebesgue integration :** Set-theoretic operations. Lebesgue measure on the line, measurable functions, Lebesgue integral, convergence almost everywhere, monotone and dominated convergence theorems, Product Spaces and Fubini's theorem.
7. **Complex analysis :** Analytic functions, Cauchy's theorem and Cauchy integral formula, maximum modulus principle, Laurent series, singularities, theory of residues, contour integration.
8. **Abstract algebra :** Groups, finite groups, symmetric and alternating groups, direct product and finite abelian groups, Sylow theorems; rings, polynomial rings, integral domains, Euclidean rings; fields, characteristic of a field, extension fields, roots of polynomials, finite fields.
9. **Ordinary differential equations :** First order ODE and their solutions, singular solutions, initial value problems for first order ODE, general theory of homogeneous and nonhomogeneous linear differential equations.

SAMPLE QUESTIONS

1. Let (X, d) be a metric space with at least two elements. Let $C(X)$ be the space of all real valued continuous functions on X. Show that $C(X)$ is not an integral domain.

2. Let X be a Hausdorff space. Let $f : X \to \mathbb{R}$ be such that $\{(x, f(x)) : x \in X\}$ is a compact subset of $X \times \mathbb{R}$. Show that f is continuous.

3. Let ℓ^2 be the Hilbert space of all square summable sequences of complex numbers with its usual inner product. Let S be the subspace consisting of sequences (x_i) such that $x_i \neq 0$ for only finitely many entries and $\sum x_i = 0$. Show that $S^\perp = \{0\}$. Here $S^\perp$ denotes the set of elements in ℓ^2 which are orthogonal to elements in S.

4. Let E and F be real or complex normed linear spaces. Let $T_n : E \to F$ be a sequence of continuous linear transformations such that $\sup_n \|T_n\| < \infty$. Let

   $$M = \{x \in E : \text{ The sequence } \{T_n(x)\} \text{ is Cauchy}\}.$$

   Show that M is a closed set.

5. Let $f : (0, 1) \to [0, 1]$ be a 1-1, continuous function. Show that f is not onto.

6. Let $p(x)$ be an odd degree polynomial in one variable with coefficients from the set $\mathbb{R}$ of real numbers. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Prove that there exists an $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

7. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Calculate eigenvalues of A and use it to find A^{100}.

8. Let L and T be two linear transformations from a real vector space V to $\mathbb{R}$ such that $L(v) = 0$ implies $T(v) = 0$. Show that $T = cL$ for some real number c.

9. Show that there are no integers x and y such that $15x^2 - 7y^2 = 9$.

10. Show that the 100 digit number $11 \cdots 1$ is divisible by the prime number 101.

11. Let m denote the Lebesgue measure on the closed interval $[0,1]$. Let A_1, A_2 and A_3 be measurable subsets of $[0,1]$ such that $m(A_i \cup A_j) = m(A_i) + m(A_j)$ for all $i \neq j$. Show that

$$m(A_1 \cup A_2 \cup A_3) = m(A_1) + m(A_2) + m(A_3).$$

Caution : A_1, A_2, A_3 need not be pairwise disjoint.

12. Let f be a uniformly continuous real valued function on the real line $\mathbb{R}$. Assume that f is integrable with respect to the Lebesgue measure on $\mathbb{R}$. Show that $f(x) \to 0$ as $|x| \to \infty$.

13. Does there exists an analytic function f on $\mathbb{C}$ such that

$$f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{2n}?$$

Justify your answer.

14. Let $\mathbb{C}$ be the set of complex numbers and let f be an analytic function on the open disc $\{z \in \mathbb{C} : |z| < 1\}$. Assume that $\left\{\frac{d^n f}{dz^n}(0)\right\}$ is a bounded sequence. Show that f has an analytic extension to $\mathbb{C}$.

15. If $\varphi : G \to H$ is a surjective group homomorphism between finite groups, show that the image of the Sylow subgroup of G is a Sylow subgroup of H.

16. Let $\mathbb{C}$ be the field of complex numbers $\varphi : \mathbb{C}[X, Y, Z] \to \mathbb{C}[t]$ be the ring homomorphism such that $\varphi(a) = a$ for all $a \in \mathbb{C}$, $\varphi(X) = t$, $\varphi(Y) = t^2$, $\varphi(Z) = t^3$. Determine the kernel of φ.

17. Let $y : [a, b] \to \mathbb{R}$ be a solution of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

where $P(x)$ and $Q(x)$ are continuous functions on $[a, b]$. If the graph of the function $y(x)$ is tangent to X-axis at any point of this interval, then prove that y is identically zero.

18. Let y_1 and y_2 be two solutions of $\frac{d^2y}{dt^2} + P(t)\frac{dy}{dt} + Q(t)y(t) = 0$ on an interval $[a, b]$ where $P(t)$ and $Q(t)$ are continuous functions on $[a, b]$. Assume that y_1 and y_2 have a common zero in $[a, b]$. Show that one of y_1, y_2 is a constant multiple of other.