Some Improved Bounds on Small Strong $\epsilon$-Nets

Snehasish Mukherjee  
M.Tech (CS), Year I  
Indian Statistical Institute, Kolkata  
Email: mtc1010@isical.ac.in

Aniket Basu Roy  
M.Tech (CS), Year I  
Indian Statistical Institute, Kolkata  
Email: mtc1006@isical.ac.in

Abstract—In this paper we present some new constructions of small strong $\epsilon$-nets for rectangles, strips and wedges based on simple geometric arguments. For rectangular range spaces we improve upon the earlier known bound on $\epsilon$ for net set size of 2. For strips and wedges for which only lower bounds were known, we bound $\epsilon$ from above for net set sizes of 2, 3 and 4. Further, we show that the net constructed for strips are also nets for wedges for net size of 3 and 4.

I. INTRODUCTION

The concept of $\epsilon$-net was introduced by Haussler and Welzel in [1] and has since found many applications in computational geometry, approximation algorithms, learning theory etc. An $\epsilon$-net for the geometric set system $(X,F)$, where $X$ is a finite set of points in $\mathbb{R}^2$ and $F$ is a non-empty family of subsets of $X$ induced by geometric objects like wedges, rectangles etc., is a set of points $N$ such that $N \cap S \neq \emptyset$ for all $S \in F$ with $|S| > \epsilon |X|$ where $0 \leq \epsilon \leq 1$. $N$ is called a strong epsilon net for the set system $(X,F)$ if $N \subseteq X$. Otherwise, i.e if $N \subseteq \mathbb{R}^2$, $N$ is called a weak $\epsilon$-net.

Following the notational conventions of epsilon net literature, we denote the range spaces defined by rectangles, strips and wedges by $R$, $T$ and $W$ respectively, the $\epsilon$-net size, i.e $|N|$, by $i$ and the value of $\epsilon$ for the range space $S$ with $|N| = i$ by $\epsilon^S_i$.

Epsilon nets have been studied for quite some time now [2], [3], [4], [5]. However, the study of small strong epsilon nets were initiated in [6]. Improved upper and lower bounds on $\epsilon^R_i$ were obtained for small values of $i$. For strips and wedges they obtained $\epsilon^T_i \geq \frac{1}{\sqrt{i+1}}$ and $\epsilon^W_i \geq \frac{1}{\sqrt{i}}$. The study of wedges was continued further in [7] where it was proved that $\epsilon$-nets of size $O(\frac{n}{\sqrt{i}})$ exists for $\alpha$-fat wedges. However no upper bounds were computed for general orientations of strips and wedges.

In this paper we present some new constructions of small strong $\epsilon$-nets for rectangles and general orientations of strips and wedges. For rectangular range spaces we show $\epsilon^R_i \geq \frac{1}{\sqrt{i}}$ which improves upon the earlier known bound in [6]. For strips in general, which had no previously known upper bounds, we show $\epsilon^T_i \leq \frac{2}{\sqrt{i}}$, $\epsilon^W_i \leq \frac{4}{\sqrt{i}}$ and $\epsilon^T_i \leq \frac{4}{\sqrt{i}}$. Further we show that for $i = 3, 4$ the nets constructed for strips are also nets for wedges. Hence $\epsilon^W_i \leq \frac{4}{\sqrt{i}}$ and $\epsilon^W_i \leq \frac{4}{\sqrt{i}}$.

II. NEW LOWER BOUND ON $\epsilon^R_i$

Let us consider the axis-parallel rectangles as range spaces, and point set $P \subseteq \mathbb{R}^2$. Take 2 points $(x_1, y_1), (x_2, y_2) \in P$ such that $\frac{3}{2} (x_3, y_3)$ where $x_1 < x_3 < x_2$. Now if the points in $P$ incident on vertical line $x = x_2$ shifted by any amount horizontally towards $x = x_1$, without coinciding or crossing, then this will not affect the set of axis-parallel rectangles containing different subsets of $P$ (see figure 1). Similar argument holds for $y$-values.

Hence, assuming that $\forall(x_1, y_1), (x_2, y_2) \in P : x_1 \neq x_2 \land y_1 \neq y_2$ we arrive at the following conclusion: For axis-parallel rectangles as range spaces, any point set $P$ of size $n$ is some permutation over $\{1, 2, ..., n\}$.

So, for $P$ of size $n$ we need to look at only $n!$ permutations instead of infinite number of point sets. Let $N \subseteq P$ be the net set of size $i$ and $R \subseteq \mathbb{R}$ be any axis-parallel rectangle. Keeping $n$ and $i$ constant, $N \in \binom{\{1, 2, ..., n\}}{i}$. We define $k$ as a function of $n$ and $i$ as follows:

$$k(n, i) = \max_{P} \min_{N} \max_{\forall R} |R : R \cap N = \emptyset|$$

and observe,

$$\epsilon^R_i = \max_{n \geq 1} \frac{k(n, i)}{n}$$

Theorem II.1. $\epsilon^R_i \geq \frac{1}{\sqrt{i}}$

Proof: Consider the point set $P$ as shown in figure 2. We claim that for any choice of net $N$, there exists a rectangle of size at least 8 that escapes points in $N$. Formally, $\forall N \exists R : R \cap N = \emptyset \land |R| \geq 8$. We try to prove its contrary i.e., $\exists N \forall R : R \cap N = \emptyset \Rightarrow |R| < 8$. So, we try to find such a $N$.

It is to be noted that $P$ is symmetric about the principal diagonal of the grid (the line joining points (1,14) and (14,1)). Every point $p_j \in P$ can be identified by its $x$-value as each has a distinct $x$ and $y$ value by our earlier assumption. So, $p_j = (j, y_j)$.

Let $N = \{p_{\nu_1}, p_{\nu_2}\} \equiv \{(\nu_1, y_{\nu_1}), (\nu_2, y_{\nu_2})\}$, where $\nu_1 < \nu_2$. Next there can be 2 cases:

1. In other words point sets are closed under vertical and horizontal strain.
2. $\nu^2$ refers to the no. of points of $P$ contained in it.
There are 8 maximal rectangles $R$ such that $R \cap N = \phi$ : 3 columns $c_1, c_2, c_3$, 3 rows $r_1, r_2, r_3$, and 2 others; left-up $(LU)$ and right-bottom $(RB)$, or left-bottom $(LB)$ and right-up $(RU)$ for the above cases respectively (see figure 3).

![Figure 2](image1.png)

**Fig. 2.** A point set in a $14 \times 14$ grid.

![Figure 3](image2.png)

**Fig. 3.** List of 8 maximal rectangles $R$ such that $R \cap N = \phi$ where $N = \{p_{v_1}, p_{v_2}\}$.

In order to have the size of the 3 columns $< 8$ the following conditions are needed to be satisfied:

1a) $v_1 \leq 8$

2a) $v_2 \geq 7$

3a) $1 \leq v_2 - v_1 \leq 8$

Likewise, for rows the following conditions are needed to be satisfied:

1b) $\min\{y_{v_1}, y_{v_2}\} \leq 8$

2b) $\max\{y_{v_1}, y_{v_2}\} \geq 7$

3b) $1 \leq |y_{v_2} - y_{v_1}| \leq 8$

*Case i. $y_{v_1} < y_{v_2}$: If conditions 1 and 2 (both a and b) are satisfied, then $v_1 \in \{1, 2, 3, 5, 7, 8\}$ and $v_2 \in \{9, 10, 11, 12, 13, 14\}$ (see figure 4, rectangles formed of solid lines).

- For $v_1 = 3$ and $v_2 = 9$, $|RB| = 8$.

![Figure 4](image3.png)

**Fig. 4.** The 2 rectangles formed of solid lines contain the values of $N$ satisfying conditions 1 and 2 for Case i and the broken ones for Case ii. Case i: $v_1 \in \{1, 2, 3, 5, 7, 8\}$ and $v_2 \in \{9, 10, 11, 12, 13, 14\}$. Case ii: $v_1 \in \{4, 6\}$ and $v_2 \in \{7, 8, 13, 14\}$.

- For $v_1 = \{1, 2, 3\}$ and $v_2 = \{9, 10, 11\}$, $|RB| \geq 8$ since for $v_1 \leq 3$ and $y_{v_2} \geq y_9$ does not make $|RB| < 8$.

- For $v_1 = 5$ and $v_2 = 12$, $|LU| = 8$.

- For $v_1 = \{5, 7, 8\}$ and $v_2 = \{12, 13, 14\}$, $|LU| \geq 8$ since (similar as above) $y_{v_1} \leq y_5$ and $v_2 \geq 12$ does not makes $|LU| < 8$.

- For $v_1 = 3$ and $v_2 = 12$, condition 3a is dissatisfied.

- For $v_1 = \{1, 2, 3\}$ and $v_2 = \{12, 13, 14\}$ also, condition 3a is dissatisfied.

- By symmetry (about the principal diagonal), similarly as above, for $v_1 = \{5, 7, 8\}$ and $v_2 = \{9, 10, 11\}$ condition 3b is dissatisfied.

*Case ii. $y_{v_1} > y_{v_2}$: If conditions 1 and 2 are satisfied, then $v_1 \in \{4, 6\}$ and $v_2 \in \{7, 8, 13, 14\}$ (See figure 4, rectangles formed of broken lines).

- For $v_1 = 4$ and $v_2 = 8$, $|RU| = 8$; and for $v_2 = 7$, $|RU| = 9$.

- For $v_1 = 6$ and $v_2 \in \{7, 8\}$ condition 3b is dissatisfied.

- By symmetry similar is the case for $v_2 \in \{13, 14\}$.

Hence, proved that there does not exists a net $N$ for which all rectangles $R$ escaping it have size $< 8$. Thus our initial claim is true.

### III. General Approach for Strips and Wedges

The general approach for obtaining upper bounds on $\epsilon$ for strips and wedges $^3$ in the plane starts by partitioning the input point set into several non-empty partitions by means of vertical, horizontal and/or Ham-sandwich cuts. Each partition is represented by its convex hull (abbr. CH). For representational purposes, the CHs are drawn as circles or ellipses. However properties of circle or ellipse never play any role in our proofs.

$^3$denoted by $\mathcal{T}$ and $\mathcal{W}$ respectively
The net is generally chosen from the set of those vertices of the CHs through which some direct common tangents (abb. DCT) between the CHs pass\(^4\). The proof technique consists of exhaustively classifying all possible strips and wedges and for each class showing that strips and wedges that escapes the net, also misses some of the CHs, and hence partitions completely.

All throughout we have made the following assumptions:

III.1 Points are in general position.
III.2 A point set can be partitioned into four regions in any desired ratio with two straight lines.
III.3 Intersection of the partitioning lines do not introduce any singularity.
III.4 Strips and wedges do not contain the points on their supporting lines.

Theorem 1 in [9] guarantees the validity of assumption III.2.

IV. SOME FREQUENTLY USED GEOMETRIC ARGUMENTS

In this section we state and prove four lemmata based on some simple geometric arguments. These will be referred to frequently while proving our main results. Though some of the lemmata in this section follow from their preceding ones, seperately stating them greatly elucidates proofs in certain cases.

We start out by re-defining the terms origin, axes and quadrants w.r.t our present context.

**Definition IV.1.** Origin is defined as the point of intersection of two straight lines in \(\mathbb{R}^2\) that partition the plane into four disjoint regions. The two straight lines are called the axes and the four disjoint regions so created are called the quadrants.

**Lemma IV.2.** If \(H\) and \(V\) be two lines intersecting at \(O\) thereby creating a set of four quadrants \(Q = \{Q_i : 1 \leq i \leq 4\}\) and \(L\) be a line that intersects both \(H\) and \(V\) in \(Q_i\) (see figure 5), then \(Q_j \cap L = \phi\) where \(Q_j \in Q\) is vertically opposite to \(Q_i\). Also \(\forall T\) (or \(W\)) such that \(O \notin T\), \(\exists Q_k \in Q\) such that \(T \cap Q_k = \phi\).

**Proof:** From basic geometry we know that two lines can intersect at only one point. If \(L\) intersects both axes (i.e. \(H\) and \(V\)) in a quadrant, it cannot intersect them (the axes) again in the vertically opposite quadrant. Hence \(L\) cannot have any portion in the vertically opposite quadrant. For proving the second part we observe that the fattest strips (read ‘strips and wedges’ hence forth in this proof) that do not contain the origin will have one supporting line, say \(M\) (see figure 5), passing over the origin \(O\). Clearly for all possible orientations of \(M\), it has no intersection with two vertically opposite quadrants that lie entirely on two different half planes defined by \(M\). Since a strip, with \(M\) as a supporting line, is a subset of one of the half planes defined by \(M\), strips not containing the origin are bound to miss one of these quadrants. \(\blacksquare\)

**Lemma IV.3.** Let \(H\), \(V_1\) and \(V_2\) be three straight lines such that \(V_1\) and \(V_2\) intersect \(H\) at \(O_2\) and \(O_1\) respectively. Let \(\text{assumption III.1. ensures no DCT can pass through more than 1 point of the same CH}^4\)
perpendicular to its length towards $O$ until $\mathcal{H}P_i \cap Q_i \neq \emptyset$, then either $P_1 \in \mathcal{H}P_i$ or $P_2 \in \mathcal{H}P_i$ or both. In other words, if $\mathcal{H}P_i$ has non empty intersection with $Q_i$ then it contains at least one of $\{P_1, P_2\}$.

\[ \text{Fig. 7. Line L passes over Q}_{12} \text{ completely before entering Q}_{i4} \]

\[ \text{Proof: Let L} \prime \text{ represent L when L is passing through O (see figure 7). Clearly, when L is over O either Q}_{i2} \subset \mathcal{H}P_i \text{ and } Q_{i4} \cap \mathcal{H}P_i = \emptyset \text{ (as is the case in figure 7) or Q}_{i2} \subset \mathcal{H}P_i \text{ and } Q_{i4} \cap \mathcal{H}P_i = \emptyset \text{ depending upon the mutual orientation of V and L} \prime \text{ (we have assumed V is not parallel to L} \prime \text{, otherwise the lemma is obviously true). Since P}_1 \in Q_{i4} \text{ and } P_2 \in Q_{i2}, \text{ either } P_1 \in \mathcal{H}P_i \text{ or } P_2 \in \mathcal{H}P_i \text{ or both, when L passes through O. This obviously remains true as L continues its translation until } \mathcal{H}P_i \cap Q_i \neq \emptyset. \]

Lemma IV.5. Let $CH_1$ and $CH_2$ be two convex hulls with DCT $T$, $P_1$ and $P_2$ be the vertices of $CH_1$ and $CH_2$ respectively that defines $T$ and $H$ be a line intersecting $T$ with $H \cap CH_i = \emptyset, \forall i \in \{1, 2\}$. If $L$ be any line intersecting $T$ with point of intersection being at or nearer to $P_1$ and if $L$ has the property that it does not intersect the segment $P_1P_2$ and does not intersect $H$ in the half plane (defined by $T$) that contains the CHs (see figure 8), then $L \cap CH_{3-i} = \emptyset$.

\[ \text{Fig. 8. Line L cannot intersect CH}_1 \]

\[ \text{Proof: T, being the DCT of CH}_1 \text{ and CH}_2, \text{ defines two half-planes, } \mathcal{H}P_1 \text{ containing none of the CHs and } \mathcal{H}P_2 \text{ containing both the CHs (see figure 8). Since L intersects T, L has one portion in } \mathcal{H}P_1 \text{ and another in } \mathcal{H}P_2. \text{ Clearly the portion of } L \text{ in } \mathcal{H}P_1 \text{ cannot intersect any of the CHs. The portion of } L \text{ in the other half plane, } \mathcal{H}P_2, \text{ can intersect } CH_i \text{ but does not intersect } H \text{ and hence can never intersect } CH_{3-i}. \]

Lemma IV.6. Let $CH_1$, $CH_2$, $T$, $P_1$, $P_2$, $\mathcal{H}P_1$, $\mathcal{H}P_2$ and $H$ be as defined in lemma IV.5 and $O$ be the point of intersection of $H$ and $T$. Let $L$ be any arbitrary line intersecting $H$ in $\mathcal{H}P_1$, and defining two half-planes, $\mathcal{H}P_i^L$ and $\mathcal{H}P_i^R$ such that $\mathcal{H}P_i^L \cap CH_i = \emptyset, i \in \{1, 2\}$. Now if $L$ is translated perpendicular to its length towards $O$ (see figure 9) until $\mathcal{H}P_i^L \cap CH_i \neq \emptyset, \forall i \in \{1, 2\}$, then either $P_1 \in \mathcal{H}P_i^L$ or $P_2 \in \mathcal{H}P_i^L$ or both. In other words, if $L$ has non empty intersection with both $CH_1$ and $CH_2$ then $\mathcal{H}P_i^L$ contains at least one of $\{P_1, P_2\}$.

\[ \text{Fig. 9. Line L cannot intersect CH}_2 \text{ without passing through P} \]

\[ \text{Proof: If } L \parallel T \text{ then the claim is obvious. Otherwise either } P_1 \text{ or } P_2 \text{ will be nearer to } L \text{ than the other. If } P_1 \text{ is nearer, as shown in figure 9, then } R, \text{ the point of intersection of } L \text{ and } T, \text{ will move towards } P_1, \text{ as the line } L \text{ moves towards } O. \text{ Thus } R \text{ will first pass through } P_1 \text{ and then } O. \text{ Therefore, when } R \text{ is at } O, \text{ as shown by the dotted line, } L \text{ has yet not entered the quadrant (defined by } T \text{ and } H) \text{ containing } CH_2 \text{ and } P_1 \in \mathcal{H}P_1^L. \text{ This will obviously hold true as } L \text{ continues translation until } \mathcal{H}P_1^L \cap CH_2 \neq \emptyset. \text{ Similarly, if } L \text{ is nearer to } P_2 \text{ than } P_1. \]

V. Upper Bound on $\epsilon^2_T$

\[ \text{Theorem I.1. } \epsilon^2_T \leq \frac{7}{8} \]

\[ \text{Proof: Partition the input point set containing } n \text{ points into four quadrants } Q_i, 1 \leq i \leq 4 \text{ by two lines } H \text{ and } V \text{ intersecting at } O_1 \text{ such that } Q_1, Q_2, Q_3 \text{ and } Q_4 \text{ contains } \frac{n}{4}, \frac{n}{4}, \frac{n}{4} \text{ and } \frac{n}{4} \text{ input points respectively as shown in figure 10. Bisect } Q_1 \text{ and } Q_4 \text{ simultaneously by the Ham-sandwich cut } H_1 \text{ which intersects } H \text{ at } O_2. \text{ Bisect the two partitions} \]
so formed in $Q_1$ again by the Ham-sandwich cut $H_2$ which intersects $H_1$ at $O_3$ and $V$ at $O_4$ (see figure 11). This creates eight partitions $R_i$, $1 \leq i \leq 8$ containing $\frac{n}{8}$ input points each. Let $CH_1$ and $CH_2$ be the CHs of the points in $R_1$ and $R_4$ respectively. Let $P_1$ and $P_2$ be the points on $CH_1$ and $CH_2$ respectively through which the DCT $T$ between them pass (see figure 12). We claim $\{P_1, P_2\}$ is a $\frac{n}{8}$-net for strips.

Fig. 10. Partitioning the input point set into 4:1:1:2 ratio by $V$ and $H$

Fig. 11. Final partitioning of the point set.

Consider the region $R_4$. All possible strips in the plane can be exhaustively classified into the following three types w.r.t $R_4$: strips that that exclude $R_4$ completely, strips that include $R_4$ entirely and strips that partially include $R_4$. Strips that exclude $R_4$ completely clearly misses the $\frac{n}{8}$ points contained in $R_4$. Strips that include $R_4$ entirely are obviously netted by $P_2$. Now we note that the quadrilateral $O_1O_2O_3O_4$ is convex and hence any strip that partially includes $R_4$ will have atleast one supporting line intersecting any two sides of the quadrilateral. However the vertices $O_1, O_2, O_3$ of the quadrilateral are origins as defined in definition IV.1 and hence from lemma IV.3 it is obvious that any strip with a supporting line intersecting either the side $O_1O_2$ or $O_2O_3$ or both must miss atleast $\frac{n}{8}$ input points. Thus, to prove theorem VI it suffices to show that all strips $T_f$ that escapes the net and has atleast one supporting line intersecting the the pair of sides $O_3O_4$ and $O_4O_1$ also misses atleast $\frac{n}{8}$ input points. We do this in the following paragraphs.

Let $\mathcal{L}$ be any arbitrary supporting line of $T_f$ that intersects the pair of sides $O_3O_4$ and $O_4O_1$ as shown in figure 13. $\mathcal{L}$ defines two half planes $\mathcal{H}P_1$ containing $O_2$ and $\mathcal{H}P_2$ containing $O_4$. The other supporting line $\mathcal{L}'$ of $T_f$ has to be on $\mathcal{H}P_1$, else the strip would miss $O_2$ and hence $\frac{n}{8}$ input points (from lemma IV.2). Also, $\mathcal{L}'$ cannot intersect the quadrilateral $O_1O_2O_3O_4$ because, like $\mathcal{L}$, $\mathcal{L}'$ will have to intersect the pair of sides $O_3O_4$ and $O_4O_1$ which implies $T_f$ misses $O_2$ and hence
work carried out during May - June, 2011

Let $\alpha$ and $\beta$ denote the angle subtended by $V$ and $L$ respectively on $T$ and $H_1^T$ and $H_2^T$ be the two half planes defined by $T$. Clearly, if $L$ is of type $VI.C$, then $\alpha \geq \beta$ (see line $L_3$ in figure 14) and hence such $L$ will intersect $O_1O_4$ in $H_1^T$. Therefore, from lemma IV.5, $T_f$ with $L$ of type $VI.C$ misses $CH_1$ completely.

Lastly, $L$ cannot simultaneously be of type $VI.D$ and intersect $V$ in $H_1^T$ since otherwise $T_f$ includes both $P_1$ and $P_2$ (see line $L_4$ in figure 14). If $L$ is of type $D$ and intersects $V$ in $H_1^T$ (see line $L_5$ in figure 14) then, from lemma IV.5, $T_f$ misses $CH_2$ completely.

Thus $L$ cannot be of any of the above types and hence all $T_f$ misses atleast $\frac{n}{8}$ input points.

Note: Removing the constraint that supporting lines are parallel does not affect most of the arguments above and it seems very likely that the bound holds for wedges too, i.e. $\epsilon_2^W \leq \frac{r}{8}$. But we know $\epsilon_1^W \geq \frac{r}{2}$ which means non-trivial upper bounds should not exist on $\epsilon_2^W$. Therefore something is amiss somewhere. Maybe assumption III.3 in section III needs to be reviewed critically.

VI. UPPER BOUND ON $\epsilon_3^T$ AND $\epsilon_3^W$ 

From now on we shall not make any assumptions about the mutual orientation of the supporting lines. Therefore proofs will be valid for both strips and wedges. Henceforth the phrase strips and wedges will be abbreviated as SAW.

**Theorem VI.1.** $\epsilon_3^T, \epsilon_3^W \leq \frac{r}{5}$

**Proof:** Partition the input point set containing $n$ points into four quadrants $Q_i$ where $1 \leq i \leq 4$, by two lines $H$ and $V$ intersecting at $O_1$ such that $Q_1, Q_2, Q_3$ and $Q_4$ contains $\frac{4n}{11}, \frac{3n}{11}, \frac{3n}{11}$ and $\frac{4n}{11}$ input points respectively as shown in figure 15. Bisect $Q_1$ and $Q_4$ simultaneously by the Ham-sandwich cut $H_1$ which intersects $H$ at $O_2$. Partition $Q_2$ and $Q_3$ in the ratio $1:2:1:2$ with the line $H_2$ which intersects $H$ at $O_3$ (see figure 16). This creates eight partitions $R_i$, $1 \leq i \leq 8$ where regions $R_3$ and $R_5$ contains $\frac{n}{14}$ points each and the remaining regions contain $\frac{n}{7}$ points each. Let $CH_1$ and $CH_2$ be the CHs of the points in regions $R_6$ and $R_7$ respectively. Let $P_1$ and $P_2$ be the points on $CH_1$ and $CH_2$ respectively through which the DCT $T$ between $CH_1$ and $CH_2$ pass (see figure 17). $P_3$ be any point in $R_2$. We claim $\{P_1, P_2, P_3\}$ is a $\frac{n}{7}$-net for SAW.

Consider the triangle $P_1P_2P_3$ (see figure 17). All possible SAW in the plane can be exhaustively classified into the following two mutually exclusive types:

VI.1.a) SAW that intersect the triangle $P_1P_2P_3$.

VI.1.b) SAW that do not intersect the triangle $P_1P_2P_3$.

In the following two lemmata, lemma VI.2 and lemma VI.3, we prove that SAW that belongs to either of these two types and escapes the net, misses at least $\frac{n}{7}$ input points, which in turn proves our claim that $\{P_1, P_2, P_3\}$ is a $\frac{n}{7}$-net.

**Lemma VI.2.** SAW of type VI.1.a that escapes the net, misses at least $\frac{n}{7}$ input points.

**Proof of Lemma VI.2:** Both supporting lines of SAW of type VI.1.a that escapes the net, must intersect the same two
sides of the triangle $P_1P_2P_3$, otherwise they end up containing a vertex of the triangle which is a hitting point. There can be three such distinct pairs and in the next three paragraphs we consider each of them.

**Supporting lines intersecting the pair of sides $P_1P_2$ and $P_2P_3$:** SAW in this case will have supporting lines $B_S_1$ and $U_S_1$ as shown in figure 18. $B_S_1$ and $U_S_1$ can be pulled apart from one another as far as possible while maintaining the condition that SAW have to escape the net. From lemma IV.4 it follows that the right supporting line $R_S$, and hence the strip or wedge with $R_S$ as a supporting line, will have empty intersection with at least one of $R_1$ or $R_8$. Thus such SAW miss at least $\frac{n}{7}$ input points.

Thus all possible SAW of type VI.1.a that escapes the net also misses at least $\frac{n}{7}$ input points.

**Lemma VI.3.** SAW of type VI.1.b that escapes the net, misses at least $\frac{n}{7}$ input points.

**Proof of Lemma VI.3:** Two cases arise for SAW of type VI.1.b that escapes the net.

**Case 1.** The triangle $P_1P_2P_3$, in figure 17, encloses the point $O_1$: In this case SAW of type VI.1.b are bound to miss the point $O_1$ and hence at least $\frac{3n}{11}$ input points (from lemma IV.2).

**Case 2.** The triangle $P_1P_2P_3$ does not enclose the point $O'$: In this case the points $P_1, O_2, P_3, O_1$ forms a convex quadrilateral (see figure 19). Two subcases arise.

**Case 2a.** SAW that do not intersect the quadrilateral $P_1O_2P_3O_1$: In this case SAW miss both $O_1$ and $O_2$ and hence at least $\frac{n}{11}$ input points (from lemma IV.2).

**Case 2b.** SAW that intersect the quadrilateral $P_1O_2P_3O_1$: Since SAW of type VI.1.b do not intersect the triangle $P_1P_2P_3$, in this case SAW must have at least one supporting line intersecting either the pair of sides $\{P_3O_1, O_1P_1\}$ or $\{P_1O_2, O_2P_3\}$ (see figure 19). In both the cases the supporting line intersects the line segment $O_1O_2$ and hence, from lemma IV.3, such SAW misses at least $\frac{n}{7}$ input points.

Clearly, from all the cases considered, SAW of type VI.1.b that escapes the net, misses at least $\frac{n}{7}$ input points.

**VII. UPPER BOUND ON $\epsilon^T_4$ AND $\epsilon^W_4$**

**Theorem VII.1.** $\epsilon^T_4, \epsilon^W_4 \leq \frac{3}{4}$
VII.3 \[ \text{Lemma VII.3. SAW of type VII.1.b that escapes the net, misses at least } \frac{n}{4} \text{ input points.} \]

**Proof of Lemma VII.3:** The quadrilateral \( P_1P_2P_3P_4 \) may be convex or concave. First we shall deal with the convex case. For SAW of type VII.1.b that escapes the net, both the supporting lines must intersect the same two edges of the convex quadrilateral \( P_1P_2P_3P_4 \), otherwise they end up containing a vertex which is a hitting point. This gives rise to \( \binom{4}{2} = 6 \) different cases based on which pair of sides does the supporting lines intersect. However, it suffices to show that SAW misses at least \( \frac{n}{4} \) input points only in the following three cases, since from symmetry considerations, the other three cases will have similar proofs.

**Case 1. SAW intersecting sides \( P_4P_1 \) and \( P_3P_2 \):** In this case SAW will have the supporting lines \( BS \) and \( US \) as shown in figure 22. \( BS \) and \( US \) can be pulled apart from another as far as possible while maintaining the condition that SAW have to escape the net. Consider the supporting line \( BS \). \( BS \) cannot intersect \( P_1P_2 \) to the left of \( P_2 \), and hence intersects \( T_2 \) at or above \( P_2 \). From **lemma IV.5 and lemma IV.4** it follows that \( BS \), and hence the strip or wedge with \( BS \) as the supporting line, will miss \( CH_3 \) and \( CH_7 \) respectively. Hence in this case, SAW misses at least \( \frac{n}{4} \) input points.

**Case 2. SAW intersecting sides \( P_1P_2 \) and \( P_3P_4 \):** In this case SAW will have supporting lines \( LS \) and \( RS \) as shown in figure 23. \( LS \) and \( RS \) can be pulled apart from one another as far as possible while maintaining the condition that SAW...
have to escape the net as shown in the figure. From lemma IV.6 it follows that \( \mathcal{LS} \) is bound to miss either \( CH_2 \) or \( CH_3 \) and \( \mathcal{RS} \) is bound to miss either \( CH_1 \) or \( CH_4 \). Hence such SAW misses at least \( \frac{n}{4} \) input points.

Fig. 23. SAW escaping the net and intersecting the pair of sides \( P_3P_1 \) and \( P_3P_3 \)

Case 3. SAW intersecting sides \( P_4P_1 \) and \( P_2P_3 \): In this case SAW will have supporting lines \( \mathcal{US} \) and \( \mathcal{BS} \) as shown in figure 24. \( \mathcal{US} \) and \( \mathcal{BS} \) can be pulled apart from one another as far as possible while maintaining the condition that SAW have to escape the net as shown in the figure. From lemma IV.4 it follows that \( \mathcal{US} \) is bound to miss either \( CH_3 \) or \( CH_6 \) and \( \mathcal{BS} \) is bound to miss either \( CH_7 \) or \( CH_4 \). Hence such SAW misses at least \( \frac{n}{4} \) input points. Now let us consider the case when quadrilateral \( P_1P_2P_3P_4 \) is concave. Assume w.l.g.\(^6\) that the concavity is due to the point \( P_3 \) as shown in figure 25. The concavity introduces only one extra case in addition to the cases considered for the convex case; that of both supporting lines \( \mathcal{LS} \) and \( \mathcal{RS} \) intersecting all the four sides of the quadrilateral as shown in figure 25. Clearly, both \( \mathcal{LS} \) and \( \mathcal{RS} \) has intersection in the second, third and fourth quadrants that are formed by \( V \) and \( H \) as axes\(^7\). From lemma IV.2 it follows that such SAW cannot enter the first quadrant. Therefore, for such point sets, SAW intersecting the quadrilateral misses at least \( \frac{n}{4} \) input points. There are no more cases to consider for SAW type VII.1.b that escapes the net. Thus such SAW misses at least \( \frac{n}{4} \) input points.

Fig. 24. SAW escaping the net and intersecting the pair of sides \( P_4P_1 \) and \( P_2P_3 \)

Fig. 25. SAW intersecting all four sides of the concave quadrilateral \( P_1P_2P_3P_4 \)

VIII. CONCLUSION

The lower bound for axis-parallel rectangles was obtained using computer simulation. As the simulation has a running time of \( O(n!) \), time requirements are prohibitive for large values of \( n \). If an equivalent but more efficient simulation can be designed, bounds for larger values of \( n \) can be explored. Also, the behaviour of the function \( k(n, i) \) can be studied further in the light of the observation that \( \forall n \forall i : 0 \leq k(n+1, i) - k(n, i) \leq 1 \).

As permutations are abstraction for point sets with respect to the axis-parallel rectangles, similar ideas may be applied to other range spaces like strips and wedges. However for strips and wedges, cartesian coordinate system seems to be of little help and alternatives need to be devised for this purpose.

Lastly, there is a considerable distance between lower and upper bounds on \( \epsilon \) for strips and wedges and as such attempts

---

\(^6\)from symmetry considerations the treatment when concavity is due to any other point will be similar

\(^7\)quadrants are numbered in counter-clockwise fashion starting from top right
to improve their lower bounds may be fruitful. Strips have been considered to be special case of wedges and therefore the parallelism of their supporting lines remains to be exploited for \( i \geq 3 \). Alternative approaches, that are fundamentally different from using common tangents to convex hulls, may also be investigated to lend a new perspective to the problem.

REFERENCES


---

\( ^8 \) a strip is a wedge with supporting lines intersecting at infinity