Testing hypotheses in the quantum world, Neyman-Pearson revisited

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This is a completely expository lecture on testing quantum statistical hypotheses for finite level systems. It is based on my study of the book [1] by Masahito Hayashi. By a state in a finite dimensional complex Hilbert space H we mean a positive operator of unit trace in H. Suppose ρ and σ are two states on H. By a test T for discriminating ρ from σ we mean a positive operator T satisfying $0 \leq T \leq I$. It is the quantum analogue of a randomized decision rule. Error probabilities $\alpha(T)$ and $\beta(T)$ of the first and second kind are defined respectively by $\alpha(T) \triangleq \mathrm{Tr}\,\rho(I-T)$, $\beta(T) = \mathrm{Tr}\,\beta T$. For any $0 < \varepsilon < 1$ define

$$\beta(\rho, \sigma, \varepsilon) = \inf_T \{\beta(T) | \alpha(T) < \varepsilon\},$$

which is the least possible value for the error probability of the second kind when T varies over all tests with error probability of the first kind not exceeding ε.

Theorem 1 (Quantum Stein's Lemma) For any $0 < \varepsilon < 1$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \beta(\rho^{\otimes^n}, \sigma^{\otimes^n}, \varepsilon) = \mathrm{Tr}\,\rho \log \rho - \mathrm{Tr}\,\rho \log \sigma.$$

Remark The right hand side in the statement of Theorem 1 is the Kulback-Leibler divergence of σ from ρ. It is an example of the Large Deviation Principle.

From Theorem 1 we shall deduce Shannon's coding theorem for classical-quantum channels in quantum information theory.

References

[1] Masahito Hayashi (2006). *Quantum Information, An Introduction.* Springer-Verlag, Berlin.