

**SELF-INTERACTING
RANDOM WALKS ON \mathbb{Z}**

Bálint Tóth

Institute of Mathematics

Budapest University of Technology

Kolkata, 13 December 2007

Notation: Discrete time setting:

$n \mapsto X(n)$ n.n. RW on \mathbb{Z} , with $X(0) = 0$.

Local time *on sites*:

$$l(n, k) := \left| \{0 \leq m \leq n : X(m) = k\} \right|$$

Local time *on unoriented edges*:

$$l_{\pm}(n, k) := \left| \{0 \leq m \leq n : \{X(m), X(m+1)\} = \{k, k \pm 1\}\} \right|$$

Local time *on oriented edges*:

$$\tilde{l}_{\pm}(n, k) := \left| \{0 \leq m \leq n : (X(m), X(m+1)) = (k, k \pm 1)\} \right|$$

Obvious identities:

$$\begin{aligned} l_+(n, k) &= \tilde{l}_+(n, k) + \tilde{l}_-(n, k+1) \\ &= l_-(n, k+1), \\ l(n, k) &= (l_+(n, k) + l_-(n, k))/2 \end{aligned}$$

Notation: Continuous time setting:

$t \mapsto X(t)$ continuous time RW on \mathbb{Z} , with n.n. jumps, with $X(0) = 0$.

Local time on sites:

$$\ell(t, k) := \left| \{0 \leq s < t : X(s) = k\} \right|$$

Notation: Continuous space and continuous time setting:

$t \mapsto X(t)$ continuous time process on \mathbb{R} , with $X(0) = 0$.

Local time, or occupation time density:

$$\ell(t, x) := \partial_x \left| \{0 \leq s < t : X(s) < x\} \right|$$

Example: Reinforced random walk, with edge reinforcement (ERRW)

[D. Coppersmith, P. Diaconis: 1987],

[B. Davis: 1990-1996],

[R. Pemantle: 1988-2007],

[F. Merkl, S. Rolles: 2004-2007],

.....

$$\begin{aligned} \mathbf{P}\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right) \\ \sim 1 + \alpha l_{\pm}(n, X(n)) \end{aligned}$$

Facts about:

- Almost surely

$$\lim_{n \rightarrow \infty} \frac{\ell(n, k)}{n} = \xi_k \in (0, 1), \quad \sum_{k \in \mathbb{Z}} \xi_k = 1.$$

The joint distribution of the occupation densities ξ_k is explicitly known.

- In particular: recurrence and tightness of $X(n)$ follows.

Example: Reinforced Random Walk, with vertex reinforcement (VRRW):

[R. Pemantle, S. Volkov: 1999],

[P. Tarrés: 2004],

.....

$$\begin{aligned} \mathbf{P}\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right) \\ \sim 1 + \alpha l(n, X(n) \pm 1) \end{aligned}$$

Facts about:

- The *asymptotic range*

$$\mathcal{AR} := \{k \in \mathbb{Z} : \lim_{n \rightarrow \infty} l(n, k) = \infty\}$$

Almost surely $\exists K \in \mathbb{Z}$ so that

$$\mathcal{AR} = \{K - 2, K - 1, K, K + 1, K + 2\}$$

Example: Continuous time reinforced random walk with vertex reinforcement (CTVRRW):

[B. Davis, S. Volkov: 2002-2004],

.....

$$\begin{aligned} \mathbf{P}\left(X(t + dt) = X(t) \pm 1 \mid \mathcal{F}_t\right) \\ = \left(1 + \alpha \ell(t, X(t) \pm 1)\right) dt + o(dt) \end{aligned}$$

Facts about:

- Almost surely

$$\lim_{t \rightarrow \infty} \frac{\ell(t, k)}{t} = \xi_k \in (0, 1), \quad \sum_{k \in \mathbb{Z}} \xi_k = 1.$$

The joint distribution of the occupation densities ξ_k is explicitly known.

- In particular: recurrence and tightness of $X(t)$ follows.

Example: “True” self-avoiding random walk with vertex repulsion (VTSAW):

[D. Amit, G. Parisi, L. Peliti: 1983],

[L. Peliti, L. Pietronero: 1987],

.....

$$P\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right)$$

$$\sim \exp\{-\beta \ell(n, X(n) \pm 1)\}$$

$$\sim \exp\{\beta(\ell(n, X(n)) - \ell(n, X(n) \pm 1))\}$$

Facts about: Only guesses:

- Recurrence (?)
- Nondegenerate scaling of $n^{-2/3}X(n)$ (with no hint about the scaling limit) (?)
RG argument
- $\dim \geq 3$: diffusive scaling (?)
RG argument

Example: “True” self-avoiding random walk with edge repulsion (ETSAW):

[B. Tóth: 1995-97],

[B. Tóth, W. Werner: 1997]

$$\begin{aligned} \mathbf{P}\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right) \\ \sim \exp\{-2\beta l_{\pm}(n, X(n))\} \\ \sim \exp\{\beta(l_{\mp}(n, X(n)) - l_{\pm}(n, X(n)))\} \end{aligned}$$

Facts about:

- Local limit theorem for

$$n^{-2/3} X(\theta_{n/s})$$

where

$$\begin{aligned} \mathbf{P}\left(\theta_{n/s} = m\right) = \\ \left(1 - \exp\{-s/n\}\right) \exp\{-(sm)/n\}. \end{aligned}$$

This means “almost”

$$\lim_{n \rightarrow \infty} \mathbf{P}\left(n^{-2/3} X(n) < x\right) = \int_{-\infty}^x \pi(y) dy,$$

with $\pi(y)$ identified.

- Scaling limit of the process:

$$n^{-2/3}X([nt]) \stackrel{?}{\Rightarrow} \mathcal{X}(t)$$

[B. Tóth, W. Werner: 1997]: Construction and analysis of the “true self-repelling process” $t \mapsto \mathcal{X}(t)$.

$$d\mathcal{X}(t) \stackrel{“=”}{=} -\partial_x \ell(t, \mathcal{X}(t))dt$$

$$\stackrel{“=”}{=} \left\{ \int_{-\infty}^{\infty} \delta'(z) \ell(t, X(t) - z) dz \right\} dt$$

Continuous space-time models ("Brownian polymers"):

[J.R. Norris, L.C.G. Rogers, D. Williams: 1987],

[R.T. Durrett, L.C.G. Rogers: 1992],

[M. Cranston, T.S. Mountford: 1996],

[T.S. Mountford, P. Tarrés: 2007]

.....

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ with suitable regularity,
fixed.

$$\begin{aligned} X(t) &= B(t) + \int_0^t \left\{ \int_0^s f(X(s) - X(u)) du \right\} ds \\ &= B(t) + \int_0^t \left\{ \int_{-\infty}^{\infty} f(z) \ell(s, X(s) - z) dz \right\} ds \end{aligned}$$

$$dX(t) = dB(t) + \left\{ \int_{-\infty}^{\infty} f(z) \ell(t, X(t) - z) dz \right\} dt$$

Facts about:

- If f is bounded and of compact support, $f \geq 0$, $f \not\equiv 0$, then

$$\lim_{t \rightarrow \infty} X(t)/t = c \in (0, \infty) \quad \text{a.s.}$$

- If $f(x) = x/(1 + |x|^{\beta+1})$, $\beta \in (0, 1)$ $\alpha := 2/(1 + \beta)$ then

$$\mathbf{P}\left(\lim_{t \rightarrow \infty} X(t)/t^\alpha = \pm c\right) = 1/2$$

Natural generalizations of the ERRW

[B. Davis: 1992],

[B. Tóth 1994-2000],

.....

Fixed weight function $w : \mathbb{Z}_+ \rightarrow (0, \infty)$ monotone \uparrow or \downarrow , $w(l) \neq \text{const.}$

$$\begin{aligned} \mathbf{P}\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right) \\ \sim w(\ell_{\pm}(n, X(n))) \\ = \frac{w(\ell_{\pm}(n, X(n)))}{w(\ell_+(n, X(n))) + w(\ell_-(n, X(n)))} \end{aligned}$$

Character:

$w(l+1) \geq w(l) :$ self-attractive trajectories

$w(l+1) \leq w(l) :$ self-repelling trajectories

Facts about:

Recurrence [Davis: 1992]:

- If $\sum_{k=1}^{\infty} w(k)^{-1} = \infty$ then a.s

$$\forall k \in \mathbb{Z} : \lim_{n \rightarrow \infty} \ell(n, k) = \infty.$$

- If $\sum_{k=1}^{\infty} w(k)^{-1} < \infty$ then a.s $\exists K \in \mathbb{Z}$ so that

$$\mathcal{AR} = \{K, K + 1\}.$$

Scaling limits [Tóth 1994-1999]:

- $w(l) = \exp\{-\beta l\}$: see later ...
- $w(l) = \exp\{-\beta l^\kappa\}$, $0 < \kappa < 1$: limit theorem for

$$n^{-(\kappa+1)/(\kappa+2)} X(\theta_{n/s})$$

- $w(l) \asymp l^{-\alpha}$, $0 < \alpha < \infty$ or $w(l) = 1 + bl^{-1} + \mathcal{O}(l^{-2})$: limit theorem for

$$n^{-1/2} X(\theta_{n/s})$$

[Diffusive but non-Gaussian.]

- $w(l) \asymp l^\alpha$, $0 < \alpha < 1$: limit theorem for

$$n^{-(1-\alpha)/(2-\alpha)} X(\theta_{n/s})$$

Natural generalizations of the ETSAW

Discrete time, edge repulsion:

[Tóth: 1996]

Weight function $w : \mathbb{Z} \rightarrow (0, \infty)$
nondecreasing, $w(l) \not\equiv \text{const.}$

$$\begin{aligned} \mathbf{P}\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right) \\ \sim w(\ell_{\mp}(n, X(n)) - \ell_{\pm}(n, X(n))) \end{aligned}$$

Continuous time, vertex repulsion

[Tóth, Vető: 2007]

Weight function: $w : \mathbb{R} \rightarrow (0, \infty)$,
non-decreasing, $w(l) \not\equiv \text{const.}$

$$\begin{aligned} \mathbf{P}\left(X(t+dt) = X(t) \pm 1 \mid \mathcal{F}_t\right) \\ = w(\ell(t, X(t) \pm 1) - \ell(t, X(t)))dt + o(dt) \end{aligned}$$

Facts about:

- Recurrence.
- Scaling limit: 1-d marginals: limit theorem for

$$n^{-2/3} X(\theta_{n/s})$$

- Scaling limit:

$$n^{-2/3} X(nt) \stackrel{?}{\Rightarrow} \mathcal{X}(t)$$

The limit process (“true self-repelling motion”)

$$t \mapsto \mathcal{X}(t)$$

constructed and analyzed, [Tóth, Werner: 1998]

ETSAW with self-repulsion on oriented edges

[B. Tóth, B. Vető: 2007]

Weight function $w : \mathbb{Z} \rightarrow (0, \infty)$
nondecreasing, $w(l) \not\equiv \text{const.}$

$$\begin{aligned} \mathbf{P}\left(X(n+1) = X(n) \pm 1 \mid \mathcal{F}_n\right) \\ \sim w(\tilde{\ell}_{\mp}(n, X(n)) - \tilde{\ell}_{\pm}(n, X(n))) \end{aligned}$$

Facts about:

- Scaling limit, 1-d marginals: limit theorem for

$$n^{-1/2}X(\theta_{n/s})$$

$$n^{-1/2}X(nt) \Rightarrow \text{UNI}[-\sqrt{t}, \sqrt{t}]$$

Idea of proof:

The model: Continuous time, vertex repulsion:

$w : \mathbb{R} \rightarrow (0, \infty)$, non-decr., non-const.

$$\begin{aligned} \mathbf{P}\left(X(t + dt) = X(t) \pm 1 \mid \mathcal{F}_t\right) \\ = w(\ell(t, X(t) \pm 1) - \ell(t, X(t)))dt + o(dt) \end{aligned}$$

An auxiliary Markov process:

$$t \mapsto (\alpha(t), \xi(t)) \in \{-1, +1\} \times \mathbb{R}$$

$$\mathbf{P}\left(\alpha(t+dt) = -\alpha(t) \mid \mathcal{F}_t\right) = w(\alpha(t)\xi(t))dt + o(dt),$$

$$\partial_t \xi(t) = \alpha(t),$$

Stationary measure:

$$\mu(\pm 1, du) = \frac{1}{2Z} e^{-W(u)} du,$$

$$W(u) := \int_0^u (w(v) - w(-v)) dv,$$

$$Z := \int_{-\infty}^{\infty} e^{-W(v)} dv < \infty.$$

Some particular stopping times:

$$\sigma^\pm(s) := \inf\left\{t : \int_0^t \mathbb{1}\{\alpha(u) = \pm 1\} du \geq s\right\}$$

$$\zeta^\pm(s) := \xi(\sigma^\pm(s))$$

The construction:

Local time on (the endpoints of) the edge $\langle k, k + 1 \rangle$, $x \in \mathbb{Z}$:

$$\tau(t, k) := \ell(t, k) + \ell(t, k + 1)$$

Its inverse:

$$\theta(s, k) := \inf\{t \geq 0 : \tau(t, k) \geq s\}$$

Let

$$\alpha_k(s) :=$$

$$\mathbb{1}\{X(\theta(s, k)) = k + 1\} - \mathbb{1}\{X(\theta(s, k)) = k\},$$

$$\xi_k(s) := \ell(\theta(s, k), k + 1) - \ell(\theta(s, k), k)$$

Fact: The processes $s \mapsto (\alpha_k(s), \xi_k(s))$, $k \in \mathbb{Z}$, are *independent copies* of the Markov process $s \mapsto (\alpha(s), \xi(s))$ started with initial states $(\alpha_k(0), \xi_k(0)) = (-\text{sgn}(k + 1/2), 0)$.

Ray-Knight approach:

Inverse local times:

$$T_{k,h} := \min\{t \geq 0 : \ell(t, k) \geq h\}$$

Local time profile at $T_{k,h}$:

$$\Lambda_{k,h}(l) := \ell(T_{k,h}, l)$$

From the basic representation it follows that

$$l \mapsto \Lambda_{k,h}(l) \quad \text{is Markov}$$

Moreover:

$$l \geq k : \quad \Lambda_{k,h}(l+1) - \Lambda_{k,h}(l) = \zeta_k^-(\Lambda_{k,h}(l))$$

$$l \leq k : \quad \Lambda_{k,h}(l-1) - \Lambda_{k,h}(l) = \zeta_{k-1}^+(\Lambda_{k,h}(l)).$$