Random Walks in a random environment

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- $C_{r,s} = \sum_{z\in\mathbb{Z}^d} z_r z_s\, \pi(z)$.

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- $$I(x) = \sup_{\theta \in \mathbb{R}^d} [\langle \theta, x \rangle - \log M(\theta)]$$

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- Random Case. $p(x, z, \omega)$.

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$$\pi(\omega, x, z) = \pi(\tau_x \omega, z)$$

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$$(\mathcal{L}^\omega u)(x) = \frac{1}{2}(\Delta u)(x) + \langle b(\omega, x), (\nabla u)(x) \rangle$$

- $b(\omega) : \Omega \to \mathbb{R}^d$ and $b(\omega, x) = b(\tau_x \omega)$ through the action $\{\tau_x\}$ of $\mathbb{R}^d$ on Ω.

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- $$\lim_{n\to\infty} \frac{S_n}{n} = m \quad \text{a.e.}$$
- $$m = \begin{cases} \frac{1-E[\frac{q}{p}]}{1+E[\frac{q}{p}]} & if\ E[\frac{q}{p}] < 1 \\ 0 & otherwise \end{cases}$$

CLT, LDP Results

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- The expectation is with respect to Q^ω or $\overline{Q}$, which could produce different limits for Ψ.

Subadditivity

- Quenched

$$Q^{\omega}[S_{k+\ell} \simeq (k+\ell)a]$$
$$\geqslant Q^{\omega}[S_k \simeq ka] \times Q^{\tau_{ka}\omega}[S_\ell \simeq \ell a]$$

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- Alternate method for the quenched case.

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$$q(z|w) = \frac{\int \pi(z) \Pi_{z'} \pi(z')^{k(w,0,z')} \beta(d\pi)}{\int \Pi_{z'} \pi(z')^{k(w,0,z')} \beta(d\pi)}$$

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- $q(z|w)$ would exist a.e R and can be compared to the corresponding conditional probabilities $r(z|w)$ under R.
- The relative entropy

$$H(R) = E^R\left[\sum_z r(z|w)\log\frac{r(z|w)}{q(z|w)}\right]$$

is then well defined.

Rate function

- The function

$$\bar{I}(a) = \inf_{\substack{R:\int z_1\,dR=a \\ R \text{ ergodic}}} H(R)$$

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- With this $\overline{I}$, the LDP holds for the averaged $\overline{Q}$.
- $I(0)$ and $\overline{I}(0)$ are equal.
- Can be calculated explicitly.

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- $$I(0) = \bar{I}(0) = -\inf_{\pi \in \mathcal{C}} \inf_{\theta \in \mathbb{R}^d} \log \sum_z e^{\langle \theta, z \rangle} \pi(z)$$
- In particular $I(0) > 0$ if and only if 0 is not in range of $\sum z\pi(z)$ as π varies over $\mathcal{C}$.

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- $$\mathcal{L}^{\omega} = \frac{1}{2}\Delta + \langle b(\tau_x\,\omega), \nabla \rangle$$
- Lift the paths $x(t)$ of the diffusion on $\mathbb{R}^d$ corresponding to $\mathcal{L}^{\omega}$ to Ω by

$$\omega(t) = \tau_{x(t)}\omega$$

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- We can not be sure of finding φ for given b.
- It is easy to find a b for given φ.

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- Relative entropy

$$E^{Q^{b',\omega}}\left[\frac{1}{2}\int_0^t \|b(\omega(s)) - \frac{\nabla\varphi(\omega(s))}{2\varphi(\omega(s))} - \frac{c(\omega(s))}{\varphi(\omega(s))}\|^2 ds\right]$$

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- $$\liminf_{t\to\infty}\frac{1}{t}\log Q^{\omega}[\frac{x(t)}{t} \simeq a] \geqslant - \frac{1}{2}\int \|b - \frac{\nabla\varphi}{2\varphi} - \frac{c}{\varphi}\|^2 \varphi dP$$

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- $$I(a) = \inf_{\substack{\nabla \cdot c = 0 \\ \int c dP = a}} \frac{1}{2} \int \|b - \frac{\nabla \varphi}{2\varphi} - \frac{c}{\varphi}\|^2 \varphi dP$$

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- Need a bound on the solution of

$$u_t = \frac{1}{2}\Delta u + \langle b, \nabla u\rangle$$

with $u(0) = \exp[\langle \theta, x\rangle]$.

- By Hopf-Cole transformation $v = \log u$ this reduces to estimating

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- construct a subsolution

$$\frac{1}{2}\nabla \cdot w + \frac{1}{2}\|\nabla w\|^2 + \langle b, w\rangle \leqslant \psi(\theta)$$

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- $$\sup_{\substack{\varphi \\ \nabla\cdot c=0}} \left[\int \langle c, \theta \rangle dP - \frac{1}{2} \int \|b - \frac{\nabla\varphi}{2\varphi} - \frac{c}{\varphi}\|^2 \varphi dP \right]$$

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- $$= \sup_{\varphi} \sup_{c} \inf_{u} \left[\int \langle c, \theta + \nabla u \rangle dP \right.$$
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- $$= \sup_{\varphi} \inf_{u} \int \left[\langle b - \frac{\nabla \varphi}{2\varphi}, \theta + \nabla u \rangle + \frac{1}{2} \|\theta + \nabla u\|^2 \right] \varphi dP$$

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- $$\sup_{c,\rho}\left[\int < \theta, c > \rho dP - \frac{1}{2}\int \|c - b\|^2 \rho dP\right]$$

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