Completeness of a combination of First-order Temporal Logic and Neighbourhood Logic

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Abstract

This paper presents a completeness result, with respect to a possible world semantics, for a combination of a first-order temporal logic and Neighbourhood Logic (NL). This logic was considered by Qiu and Zhou in [14] to define semantics of a real-time OCCAM-like programming language.

Keywords: First-order temporal logic, interval logic, neighbourhood logic, possible worlds semantics, completeness.

Completeness of a Combination of Neighbourhood Logic and Temporal Logic

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1 Introduction

Various real-time logics have been proposed to reason about real-time systems. Some of these formalisms interpret formulas over interval of time [6, 12, 16, 17, 19]. Among such interval logics, Interval Temporal Logic (ITL) [12] and more specifically the Duration Calculus (DC) [19] have been used to design real-time systems. DC is an extension of ITL in the sense that the temporal variables are written in the form of the integral of state variables. The modality *chop* "$\frown$"of ITL is a contracting modality and cannot specify unbounded liveness and fairness properties of computing systems, since they are properties of the system outside any given intervals.

In [18], Zhou and Hansen proposed a first-order interval logic called Neighbourhood Logic (NL) which has provisions of specifying liveness and fairness as well as formalising real analysis. This logic has two expanding modalities $\Diamond_r$ and $\Diamond_l$, called the *left* and *right* neighbourhood modality respectively. Various notions of real analysis are also defined in [18] and this demonstrates the possibility of constructing a formal theory to specify and reason about hybrid systems. It has been shown [4, 3] that NL forms a complete first-order interval logic.

NL has been extended in several directions. In [15], DC based on NL has been proposed and the relative completeness of DC/NL was established. More significantly, two-dimensional neighbourhood logics have been introduced. In one case [14], a combination of first-order temporal logic(FTL) and NL is used to specify real-time semantics of an OCCAM-like programming language. This logic deals with the super-dense computational model which provides an abstraction of real-time behaviour of computing systems. Under super-dense behaviour, there is a sequence of continuously evolving phases and at the common point of two successive phases, there is a sequence of discrete actions. Execution of these actions take no or negligible

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time– these actions are taken to be instantaneous but ordered. Also, many real-time systems are analysed under the assumption that computation and communication are instantaneous and hence take no time. When behaviours under this are considered, it is very natural to assume that the system passes through a number of states within zero time. Such assumptions introduce additional structure in the time domain. To capture such behaviour, many models have been proposed(cf.[9, 11, 13, 8, 14]).

Under super-dense model, Qiu and Zhou in [14] assumed countably many time axes indexed by natural numbers. Real-time behaviour is divided into two classes as far as their time features are concerned– staying on an axis with continuous evolution of time or jumping instantaneously to another time axis with change of state. The logic required for this two-dimensional time space is first, an interval logic (specifically, NL) for specifying behaviours on single time axis and secondly, temporal logic for travelling between axes.

The above logic uses two type of modalites: $\Diamond_r$ and $\Diamond_l$ for specifying behaviours and for accessing intervals in the horizontal direction for each continuously changing world; and modalities $\bigcirc$, $\mathcal{W}$ of temporal logic in the vertical direction for specifying change of axis to describe the static behaviour. A proof system and some explanation and justification for the axioms connecting the two are given in ([14]) [1]. The superdense model seems to be a natural choice and the preceding logic is used to define the timed semantics of a CSP-type parallel language and specify various timed properties.

A natural extension of this logic would be to consider neighbourhood logic in both dimensions. Such a logic, called NL^2, was considered and shown to be complete in [4].

In this article, we introduce the semantics and the proof system of logic FTL-NL from [14]. The question of the completeness of this logic is then discussed. Although completeness of this logic w.r.t. the standard semantics *may* not be possible in view of Abadi's result [1], a weak form of completenes w.r.t. possible world sematics (as in [1]) is shown.

2 Two Dimensional Temporal Logic FTL-NL

Syntax and Semantics

The language $\mathcal{L}$ consists Global Variables: $\mathcal{V}\hat{=}\{x, y, z, ...\}$; Temporal Variables: $\mathcal{T}\hat{=}\{\ell, v, v_1, v_2, v_3, ...\}$, where ℓ is the length of an interval; Functions: $\mathbb{F}\hat{=}\{0, +, -, ...\}$; Predicates: $\mathbb{H}\hat{=}\{\geq, ...\}$; the usual Connectives and Quantifiers and the Modalities: $\Diamond_l$, $\Diamond_r$, $\bigcirc$ and $\mathcal{W}$.

[1] All UNU/IIST Technical Reports can be accessed at **URL** *http://www.iist.unu.edu*

Terms are generated by the following abstract syntax:

$$\theta ::= x|v|\ell|f(\theta_1,\ldots,\theta_n)|\bigcirc\theta,$$

where f is an n-ary function symbol and $\bigcirc$ is the *next* operator of temporal logic.

Formulas are generated by the following abstract syntax:

$$\phi ::= G(\theta_1,\ldots,\theta_n)|\neg\phi|\phi_1\vee\phi_2|\exists x.\phi|\bigcirc\phi|\phi_1\mathcal{W}\phi_2|\diamond_l\phi|\diamond_r\phi;$$

where $\mathcal{W}$ is the *unless* modality([10]).

A term or a formula is *global* if it has no temporal symbols; otherwise it is said to be *flexible.*

For the semantics, we fix a countably infinite family of the real line $I\!R$ indexed by the natural numbers. The interval modalities are used to define accessibility between intervals on a single axis while the modalities of temporal logic are used for accessibily between axes. The global variables are assigned meaning via a *valuation* $\nu:\mathcal{V}\rightarrow I\!R$.The meaning of the temporal variables, function and predicate symbols, given an interval $[a,b]$, is via an *interpretation function* $\mathcal{I}$ such that

1. $\mathcal{I}(k,0,[a,b])=0$
2. $\mathcal{I}(k,\ell,[a,b])=b-a$
3. $\mathcal{I}(k,v,[a,b])\in I\!R;$ for $v\in\mathcal{T}$
4. $\mathcal{I}(k,f,[a,b])=\underline{f}$, for an n-ary function f, where $\underline{f}:I\!R^n\rightarrow I\!R$ is the standard interpretation of f, where f is one of $0,+,-,\ldots$.
5. $\mathcal{I}(k,G,[a,b])=\underline{G}$, for an n-ary predicate symbol G, where $\underline{G}:I\!R^n\rightarrow\{tt,ff\}$ is the standard interpretation of G, if G is one of $=,\geq,\ldots$.

Given a valuation ν, the terms are interpreted via the function $\mathcal{I}^\nu$ by induction in the usual way such that $\mathcal{I}^\nu(k,\bigcirc\theta,[a,b])=\mathcal{I}^\nu(k+1,\theta,[a,b])$. In other words, the interpretation of $\bigcirc\theta$ at stage k, given the interval $[a,b]$ is the interpretation of θ at stage $k+1$.

A *model* is a pair $\mathcal{M}\stackrel{\text{def}}{=}<I\!I,\mathcal{I}>$, where $I\!I$ is the set of all real intervals. We shall call $\mathcal{M}_k\stackrel{\text{def}}{=}<I\!I,\mathcal{I}_k>$ a *model* of the logic at stage k, where $\mathcal{I}_k(i,.,.)=\mathcal{I}(k+i,.,.)$ and $\mathcal{M}_0=\mathcal{M}$. The meaning of the formulas is defined in the traditional way via *satisfiability* and *validity.* Let

$$\mathcal{M}_k,\nu,[a,b]\models A$$

denote that the formula A is satisfied in the interval $[a,b]$ under the model $\mathcal{M}_k$ and valuation ν. Satisfaction can then be defined as

1. $\mathcal{M}_k, \nu, [a,b] \models G(t_1, \ldots, t_n)$ iff $\underline{G}(\mathcal{I}_k^\nu(t_1,[a,b]), \ldots, \mathcal{I}_k^\nu(t_n,[a,b])) = tt$
2. $\mathcal{M}_k, \nu, [a,b] \models \neg A$ iff $\mathcal{M}_k, \nu, [a,b] \not\models A$
3. $\mathcal{M}_k, \nu, [a,b] \models A \vee B$
   iff $\mathcal{M}_k, \nu, [a,b] \models A$ or, $\mathcal{M}_k, \nu, [a,b] \models B$
4. $\mathcal{M}_k, \nu, [a,b] \models \exists x A$
   iff for some valuation ν' $x - equivalent$ to ν,
   $$\mathcal{M}_k, \nu', [a,b] \models A,$$
   where we say that ν' is $x - equivalent$ to ν if $\nu'(y) = \nu(y)$ for all $y \neq x$
5. $\mathcal{M}_k, \nu, [a,b] \models \Diamond_l A$
   iff there exists $c, a \geq c$ such that $\mathcal{M}_k, \nu, [c,a] \models A$
6. $\mathcal{M}_k, \nu, [a,b] \models \Diamond_r A$
   iff there exists $c \geq b$ such that $\mathcal{M}_k, \nu, [b,c] \models A$
7. $\mathcal{M}_k, \nu, [a,b] \models \bigcirc A$ iff $\mathcal{M}_{k+1}, \nu, [a,b] \models A$
8. $\mathcal{M}_k, \nu, [a,b] \models A \mathcal{W} B$
   iff for all $j \geq k, \mathcal{M}_j, \nu, [a,b] \models A$ or $\exists i, k \leq i \leq j$ such that $\mathcal{M}_i, \nu, [a,b] \models B$

The definition of *satisfiability* and *validity* is standard. The other modalities of temporal logic or of NL are defined as usual. For instance, $\Diamond$ (eventually) is obtained as $\Diamond\phi := \neg(\neg\phi \mathcal{W} ff)$ and $\mathcal{U}$ (until) is given by $\phi \mathcal{U} \psi = \Diamond\psi \wedge \phi \mathcal{W} \psi$.

The Proof System

The proof system consists of all the axioms and rules of inferences of the first-order temporal logic FTL [1] and NL [18].

The axioms of FTL are, together with all axioms of Propositional Temporal Logic and equality axioms, the following (cf.[1]).

T1

$$A \Leftrightarrow \bigcirc A,$$

provided A is global i.e. does not contain any temporal symbols.

T2

$$\forall x.A \Rightarrow A[x/t],$$

provided the substitution x/t does not create any new bound occurrences of variables or any new occurrences of *flexibles* terms in the scope of modal operators.

T3

$$\forall x. \bigcirc A \Leftrightarrow \bigcirc \forall x.A$$

T4

$$\forall x(A\mathcal{W}B) \Leftrightarrow (\forall x.A)\mathcal{W}B,$$

provided x is not free in B.

The axioms of NL are the following. Here $\diamond_1$ can be instantiated with $\diamond_l$ or $\diamond_r$. The following abbreviation will be adopted.

$$\begin{array}{lcl} \bar{\diamond}_1 & \stackrel{\text{def}}{=} & \begin{cases} \diamond_r, & \text{if } \diamond_1 = \diamond_l \\ \diamond_l, & \text{if } \diamond_1 = \diamond_r \end{cases} \\ \Box_1 & \stackrel{\text{def}}{=} & \neg\diamond_1\neg \\ \bar{\Box}_1 & \stackrel{\text{def}}{=} & \neg\,\bar{\diamond}_1\,\neg \\ \diamond_1^c & \stackrel{\text{def}}{=} & \bar{\diamond}_1\,\diamond_1 \end{array}$$

A1 Global formulas are not connected to intervals.

$\diamond_1 A \;\Rightarrow\; A,$ provided A is a global formula.

A2 Interval length is non-negative.

$\ell \geq 0$

A3 Neighbourhood can be of arbitrary length.

$x \geq 0 \;\Rightarrow\; \diamond_1(\ell = x)$

A4 Neighbourhood modalities can be distributed over disjunction and existential quantifier.

$$\begin{array}{lcl} \diamond_1(A \vee B) & \Rightarrow & \diamond_1 A \;\vee\; \diamond_1 B \\ \diamond_1 \exists x.A & \Rightarrow & \exists x.\diamond_1 A \end{array}$$

(The second part of **A4** implies that the analogue of Barcan Formula is true.)

A5 A left (right) neighbourhood coincides with any other left (right) neighbourhood provided they have the same length. In other words, neighbourhood is determined by its length.

$\diamond_1((\ell = x) \wedge A) \;\Rightarrow\; \Box_1((\ell = x) \Rightarrow A)$

A6 Left (right) neighbourhoods of an interval always start at the same point.

$\diamond_1\,\bar{\diamond}_1\,A \;\Rightarrow\; \Box_1\,\bar{\diamond}_1\,A$

A7 Left (right) neighbourhood of the ending (beginning) point of an interval is the interval itself, if it has the same length as the interval.

$$(\ell = x) \;\Rightarrow\; (A \iff \Diamond_1^c((\ell = x) \wedge A))$$

A8 Two consecutive left (right) expansions can be replaced by a single left (right) expansion, if the third expansion has a length of the sum of the first two.

$$((x \geq 0) \wedge (y \geq 0)) \;\Rightarrow\; (\Diamond_1((\ell = x) \wedge \Diamond_1((\ell = y) \wedge \Diamond_1 A)) \iff \Diamond_1((\ell = x + y) \wedge \Diamond_1 A))$$

In addition ,the following valid formulas are also taken as axioms.

1. Length of interval is the same at all stages.
   (LT1) $(\ell = x) \Leftrightarrow \bigcirc(\ell = x)$

2. $\bigcirc$ commutes with the neighbourhood modalities.
   (NN1) $\Diamond_l \bigcirc \phi \Leftrightarrow \bigcirc \Diamond_l \phi$
   (NN2) $\Diamond_r \bigcirc \phi \Leftrightarrow \bigcirc \Diamond_r \phi$

3. Neighbourhood modalities distributes over $\mathcal{W}$
   (WN1) $\Diamond_l(\phi_1 \mathcal{W} \phi_2) \Rightarrow (\Diamond_l \phi_1)\mathcal{W}(\Diamond_l \phi_2)$
   (WN2) $\Diamond_r(\phi_1 \mathcal{W} \phi_2) \Rightarrow (\Diamond_r \phi_1)\mathcal{W}(\Diamond_r \phi_2)$

One can easily check the validity of NN1–WN2 with respect to the above semantics(cf[14]). Note that $\mathcal{U}$ in [1] stands for the *unless* modality $\mathcal{W}$ of [10].

Non-standard Completeness

We shall now consider the question of completeness of this logic. We do not know whether the above logic will be complete w.r.t. the (natural) semantics we have already considered. Since FTL is not complete w.r.t. the standard semantics(cf. [1]), it is quite likely that the above logic is also not complete. So, let us first consider the more general possible world semantics with a weaker notion of validity. First replace $\mathbb{R}$ by a more general time domain $\mathbb{D}$ satisfying the usual axioms for $=, +, \geq$ and $-$ as in [4, 5] and the set of intervals $\mathbb{I}$ by an arbitrary set W (of possible worlds). We also have two binary relation R_l and R_r on W and $w_1 R_l w_2$ (resp. $w_1 R_r w_2$) stands for the fact that the world w_2 is immediately to the *left* (resp. *right*) of the world w_1. Call such a model $\mathcal{K} =< W, R_l, R_r, \mathbb{D}, \mathcal{I} >$ a *standard* Kripke model. The semantics can now be defined as above and so also the notion of validity wrt this semantics[4, 3].

With this semantics, one may ask whether every valid formula ϕ is provable in the logic. Such a completeness result may be too demanding. The relation of satisfiabilty may turn out to be highly complex(say,Π^1_1-complete) as was with the case of FTL as shown by Abadi [1].However, one can further weaken the notion of validity by introducing *non-standard* Kripke models [1] (non-standard,as opposed to the standard one). A formula ϕ will now be valid if it is valid in more models.

A non-standard Kripke model is obtained from the standard one by introducing two additional binary relations R_1 and R_2 on the set of worlds W such that $R_1(w_1, w_2) \wedge R_1(w_1, w_3) \rightarrow w_2 = w_3$. *These relations stand for the accessibility between worlds at different stages.* (Here, we simply consider a single world instead of countably many copies.) Intuitively, R_1 corresponds to *next* and R_2 to *eventually.*

The semantics is the same as above with only the following modifications. $\mathcal{I}^\nu(\bigcirc\theta, w) = \mathcal{I}^\nu(\theta, w_1)$, for some w_1 with $R_1(w, w_1)$.

While defining the semantics we only need to have the following restriction. The interpretation function $\mathcal{I}$ satisfies

1. For any rigid symbol s, $\mathcal{I}(s, w_1) = \mathcal{I}(s, w_2)$ i.e. its interpretation is the same in all worlds and at all stages;

2. $\mathcal{I}(\ell, w_1) = \mathcal{I}(\ell, w_2)$ for all w_1, w_2 such that $w_1 R_1 w_2$ i.e. given a world w_1, the length is the same in the world at the next stage.

We illustrate the semantics with some modal operators. Note that now $\mathcal{K} =< W, R_l, R_r, R_1, R_2, I\!\!D, \mathcal{I} >$

1. $\mathcal{K}, \nu, w \models \Diamond_l A$
   iff there exists w', wR_lw' such that $\mathcal{K}, \nu, w' \models A$

2. $\mathcal{K}, \nu, w \models \Diamond_r A$
   iff there exists w', wR_rw' such that $\mathcal{K}, \nu, w' \models A$

3. $\mathcal{K}, \nu, w \models \bigcirc A$ iff there exists w', wR_1w' such that $\mathcal{K}, \nu, w' \models A$

4. $$\mathcal{K}, \nu, w_0 \models A\mathcal{W}B \quad \text{iff}$$
   $$\forall w_1[w_0 R_2 w_1 \Rightarrow (\mathcal{K}, \nu, w_1 \models A \vee \exists w_2(w_0 R_2 w_2 \wedge w_2 R_2 w_1 \wedge \mathcal{K}, \nu, w_2 \models B))].$$

Thus $\bigcirc A$ is true if A is true in the next world and $A\mathcal{W}B$ is true if either A is true henceforth, or A is true *until* B is true.

It is not very difficult to see that a standard Kripke model can be regarded as a non-standard model also. Thus a formula will now be valid if it is valid in more models. We now state our non-standard completeness result.

THEOREM 2.1. *If Γ is a consistent set of sentences then there is a non-standard Kripke model that satisfies Γ.*

Consequently, if a formula ϕ is valid then it is provable.

Proof : The proof is a combination of the techniques in [1] and [4, 3]. The model is constructed by taking the worlds to be maximal consistent sets(msc) of sentences with *witnesses* in $\mathbb{B}$, a set of new constants introduced, together with a sentence.

Formally, let $\mathbb{B}$ be a countably infinite set of symbols not occuring in the language and let $\mathbb{T}^+$ be the extended theory obtained by adding the symbols of $\mathbb{B}$ as global constants. Let Γ^* be a maximal consistent set of sentences in the extended theory $\mathbb{T}^+$ such that $\Gamma^* \supseteq \Gamma$ and has witnesses in $\mathbb{B}$ (cf.[5]),*i.e.* whenever a sentence $\exists x.\phi \in \Gamma^*$ then there is a constant $a \in \mathbb{B}$ such that $\phi(a) \in \Gamma^*$. Let Σ be the set of global formulas of Γ^*.

We shall now construct a Kripke model $\mathcal{K} =< W, R_l, R_r, R_1, R_2, \mathbb{D}, \mathcal{I} >$ for Γ.

Let
$W = \{< \Delta, \phi >: \phi$ is a formula and Δ is an *mcs* with witnesses in $\mathbb{B}$ and $\Delta \supseteq \Sigma\}$.

Clearly, W is non-empty. The accessibility relations R_l, R_r, R_1, R_2 are defined as follows.

$$R_l(< \Delta_1, \phi_1 >, < \Delta_2, \phi_2 >) \stackrel{\text{def}}{\Longleftrightarrow} \Diamond_l \Delta_2 \subseteq \Delta_1$$

and

$$R_r(< \Delta_1, \phi_1 >, < \Delta_2, \phi_2 >) \stackrel{\text{def}}{\Longleftrightarrow} \Diamond_r \Delta_2 \subseteq \Delta_1.$$

$$R_1(< \Delta_1, \phi_1 >, < \Delta_2, \phi_2 >) \stackrel{\text{def}}{\Longleftrightarrow} \phi_1 = \phi_2 \wedge \Delta_2 = \{\phi : \bigcirc\phi \in \Delta_1\}.$$

(Note that it is easy to check that $\{\phi : \bigcirc\phi \in \Delta_1\}$ is an msc with witnesses in $\mathbb{B}$.)

$$R_2(< \Delta_1, \phi_1 >, < \Delta_2, \phi_2 >) \stackrel{\text{def}}{\Longleftrightarrow} \phi_2 \notin \Delta_1, \phi_2 \notin \Delta_2$$
$$\text{and for all } \phi,\ \phi \mathcal{W} \phi_2 \in \Delta_1 \text{ implies } \phi \in \Delta_2.$$

The domain $\mathbb{D}$ is defined as follows. In $\mathbb{B}$ define a relation $\equiv$ by

$$a \equiv b \text{ iff } a = b \in \Sigma.$$

It is easy to see that $\equiv$ is an equivalence relation on $\mathbb{B}$. Let

$$\mathbb{D} = \{[b] : b \in \mathbb{B}\}$$

be the set of equivalence classes, where $[a]$ denotes the equivalence class containing a.

The interpretation function $\mathcal{I}$ is defined as follows.

1. If s is a temporal variable or a constant, then $\mathcal{I}(s, < \Delta, \phi >) = [a]$ iff $s = a \in \Delta$
2. If f is an n-ary function symbol, then
$$\mathcal{I}(f, < \Delta, \phi >)([b_1], \ldots, [b_n]) = [c] \text{ iff } f(b_1, \ldots, b_n) = c \in \Delta.$$
3. If G is an n-ary predicate symbol, then
$$\mathcal{I}(G, < \Delta, \phi >)([b_1], \ldots, [b_n]) = tt \text{ iff } G(b_1, \ldots, b_n) \in \Delta$$

$\mathcal{I}$ is well-defined and it is not hard to check that $\mathcal{I}(f, < \Delta, \phi >)$ defines a total function. Finally, one can easily check that for rigid $f, \mathcal{I}(f, < \Delta_1, \phi_1 >) = \mathcal{I}(f, < \Delta_2, \phi_2 >)$, where $< \Delta_1, \phi_1 >$ and $< \Delta_2, \phi_2 >$ are any two worlds of W. This follows from the fact that $f(b_1, \ldots, b_n) = c$ is rigid and hence is in Σ iff it is in Δ_1 (respectively, Δ_2). Similar arguments hold for global predicate symbol G. This completes the description of the Kripke model. The following results will be useful to prove the completeness theorem.

PROPOSITION 2.1. *For any term $t(x_1, \ldots, x_n)$, where the global variables occurring in t are among $x_1, \ldots, x_n$; for any world $< \Delta, \phi >$ and valuation ν, $\mathcal{I}^\nu(t, < \Delta, \phi >) = [c]$ iff $t(b_1, \ldots, b_n) = c \in \Delta$,*
where $\nu(x_i) = [b_i]$, $1 \leq i \leq n$.

Proof : By induction on the length of the terms.

$\square$

LEMMA 2.1. *For any formula $A(x_1, \ldots, x_n)$, where the free global variables in A are among $x_1, \ldots, x_n$, for any world $< \Delta, \phi > \in W$ and valuation ν,*

$$\mathcal{K}, \nu, < \Delta, \phi > \models A(x_1, \ldots, x_n) \text{ iff } A(b_1, \ldots, b_n) \in \Delta,$$

where $\nu(x_i) = [b_i]$; $1 \leq i \leq n$.

Proof: We prove the lemma by induction on formulas. If A is a propositional letter or an atomic formula, then the result follows from the definition of $\mathcal{I}$ and Proposition 2.1..
The cases for $\neg, \vee$ and $\exists$ can be handled easily. The case $A := \Diamond_1 B$ can be shown as in [3]. We shall illustrate this with case $A := \Diamond_l B$. First assume that

$$\mathcal{K}, \nu, < \Delta, \phi > \models \Diamond_l B(x_1, \ldots, x_n)$$

Then there exists $< \Delta', \phi' >$ such that $\Diamond_l \Delta' \subseteq \Delta$ and

$$\mathcal{K}, \nu, < \Delta', \phi > \models B(x_1, \ldots, x_n).$$

By induction hypothesis, $B(b_1, \cdots, b_n) \in \Delta'$ and hence $\Diamond_l B \in \Diamond_l \Delta' \subseteq \Delta$.

Conversely, suppose

$$\Diamond_l B \in \Delta.$$

We have to show that there is a world $< \Delta_1, \phi >$ such that $\Diamond_l \Delta_1 \subseteq \Delta$ and

$$\mathcal{K}, \nu, < \Delta_1, \phi > \models B(x_1, \ldots, x_n) \text{ i.e. } B(b_1, \cdots, b_n) \in \Delta_1$$

First observe that

$$\vdash B(b_1, \cdots, b_n) \Rightarrow B(b_1, \cdots, b_n) \wedge \exists x(\ell = x)$$

and consequently

$$\vdash \Diamond_l B(b_1, \cdots, b_n) \Rightarrow \Diamond_l (B(b_1, \cdots, b_n) \wedge \exists x(\ell = x)).$$

Hence, in view of our assumption,

$$\Diamond_l \exists x(B(b_1, \cdots, b_n) \wedge \ell = x) \in \Delta$$

and so by **A 4**

$$\exists x \Diamond_l (B(b_1, \cdots, b_n) \wedge \ell = x) \in \Delta$$

Hence there is a witness $c \in \mathbb{B}$ such that

$\Diamond_l(B(b_1, \cdots, b_n) \wedge \ell = c) \in \Delta$

It is not very hard to see that $\Gamma_0 = \{B(b_1, \cdots, b_n), \ell = c\}$ is consistent. It is now a matter of technicality to obtain a maximal consistent set $\Delta_1 \supseteq \Gamma_0$ such that $\Diamond_l \Delta_1 \subseteq \Delta$, $\Delta_1 \supseteq \Sigma$ and Δ_1 has witnesses in $\mathbb{B}$.

The remaining case will be proved by induction on the depth of the modal operators $\bigcirc, \mathcal{W}, \mathcal{P}$ where the *precedes* operator $\mathcal{P}$ is defined as $\phi \mathcal{P} \psi \equiv \neg(\neg\phi \mathcal{W} \psi)$.

1^o Suppose $\bigcirc B(b_1, \ldots, b_n) \in \Delta$. Then $B(b_1, \ldots, b_n) \in \Delta^+$, where $\Delta^+ = \{A : \bigcirc A \in \Delta\}$. By induction hypothesis,

$$\mathcal{K}, \nu, < \Delta^+, \phi > \models B(x_1, \ldots, x_n)$$

and so

$$\mathcal{K}, \nu, < \Delta, \phi > \models \bigcirc B(x_1, \ldots, x_n).$$

2^o Suppose $B \mathcal{P} C \in \Delta$. Let $\Delta^+ = \{B\} \bigcup \{A' : A' \mathcal{W} C \in \Delta\}$. Δ^+ is consistent. Let $\Delta' \supseteq \Delta^+$ be an msc with witnesses in $\mathbb{B}$. We claim that

$$< \Delta, \phi > R_2 < \Delta', C > .$$

(i) Since $\vdash B \mathcal{P} C \Rightarrow \neg C$, we have $\neg C \in \Delta$ and so $C \notin \Delta$.
(ii) Now,$\vdash \neg C \mathcal{W} C$ and so $\neg C \mathcal{W} C \in \Delta$. Hence $\neg C \in \Delta^+ \subseteq \Delta'$ *i.e.* $C \notin \Delta'$.
(iii) Suppose $\psi \mathcal{W} C \in \Delta$. Then $\psi \in \Delta^+ \subseteq \Delta'$ by definition.

Let $< \Delta_1, \phi_1 >$ lie between $< \Delta, \phi >$ and $< \Delta', C >$ (wrt R_2). By definition, $C \notin \Delta_1$. This shows (by induction hypothesis) that

$$\mathcal{K}, \nu, < \Delta, \phi > \models B \mathcal{P} C.$$

3^o Suppose $B \mathcal{W} C \in \Delta$. Fix $< \Delta', \phi' >$ such that

$$< \Delta, \phi > R_2 < \Delta', \phi' > .$$

We want to show that *either* $B(b_1, \cdots, b_n) \in \Delta'$ *or* there is $< \Gamma, D >$ between $< \Delta, \phi >$ and $< \Delta', \phi' >$ such that $C(b_1, \cdots, b_n) \in \Gamma$. This will prove, by induction hypothesis, that

$$\mathcal{K}, \nu, < \Delta, \phi > \models B \mathcal{W} C.$$

Suppose $B(b_1, \cdots, b_n) \notin \Delta'$. Let

$$\Delta^+ = \{C(b_1, \cdots, b_n)\} \bigcup \{B' : B' \mathcal{W}(\neg\phi \wedge \neg C) \in \Delta\}.$$

Let $\Gamma \supseteq \Delta^+$ be an msc with witnesses in $\mathbb{B}$. Let $D = \neg\phi \wedge \neg C$. Then, as in [1], one can show that $< \Gamma, D >$ lies between $< \Delta, \phi >$

and $< \Delta', \phi' >$. Also, by definition, $C(b_1, \cdots, b_n) \in \Gamma$. Thus we have shown that in this case

$$\mathcal{K}, \nu, < \Delta, \phi > \models B\mathcal{W}C.$$

The converse implication can now be handled by appealing to the previous cases (cf.[1]). This completes the proof of the lemma.

The theorem now follows immediately since, for any ν, ϕ , by the Lemma, $\mathcal{K}, \nu, < \Gamma^*, \phi >$ is a (non-standard) model of Γ.

□

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References

[1] Martin Abadi: The power of temporal proofs. In *Theoretical Computer Science* 65, 35-83, Elsevier, 1989.

[2] R. Alur, T.A. Henzinger: Real-time logics: Complexity and expressiveness. in *Information and Computation* 104(1): pp. 35–77, May 1993.

[3] Rana Barua, Suman Roy and Zhou Chaochen: Completeness of Neighbourhood Logic. In *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science*, LNCS Vol 1563, pp 521-530, 1999.
Also in *Journal of Logic and Computation*, Vol 10 No.2, pp 271-295, 2000.

[4] Rana Barua and Zhou Chaochen: Neighbourhood Logics: NL and NL^2. *UNU/IIST Report No. 120*, August 1997.

[5] B. Dutertre: *Complete Proof Systems for First Order Interval Logic*, Tenth Annual IEEE Symp. on Logic in CS, 36-43, IEEE press 1995.

[6] J. Halpern and Y. Shoham: A Propositional Modal Logic of Time Intervals. In *Proceedings of the First IEEE Symposium on Logic in Computer Science*, pp. 279-292, Computer Society Press, 1986.

[7] Michael Hansen and Zhou Chaochen: Duration Calculus: Logical Foundations. In *Formal Aspects of Computing*, Vol 9 pp 283-330, 1997.

[8] Z.Liu, A.P.Raven and X.Li: Duration Properties of Timed Trasition Systems. *Tech Report 1997/12*, Leicester University, 1997.

[9] O.Maler, Z.Manna and A.Pneuli: From Timed to Hybrid Systems. In *Real Time: Theory in Practice*, LNCS Vol 600, 1992.

[10] Z. Manna and A. Pnueli: *The Temporal Logic of Reactive and Concurrent Systems: Specification*. Springer-Verlag, 1991.

[11] Z.Manna and A.Pneuli: Models of Reactivity. In *Acta Informatica* Vol 30 No. 7 pp 609-678, 1993.

[12] B. Moszkowski: A Temporal Logic for Multilevel Reasoning about Hardware. In *IEEE Computer* 18(2) pp 10-19, 1985.

[13] P.K.Pandya and D.Van Hung: Duration Calculus of Weakly Monotonic Time.*UNU/IIST Report No. 122*, 1997.

[14] Qiu Zongyan and Zhou Chaochen : A Combination of Interval Logic and Linear Temporal Logic. *UNU/IIST Report No. 123*, September 1997.
In *Proc. PROCOMET'98*, D.Gries and W-P de Roever(eds.), Chapman and Hall, pp 444-461, 1998.

[15] Suman Roy and Zhou Chaochen : Notes on Neighbourhood Logic. *UNU/IIST Report No. 97*, February 1997.

[16] Y. Venema: Interval Tense Logic. In *Notre Dame Journal of Formal Logic, Vol. 31, No. 4*, pp. 529-547, 1990.

[17] Y. Venema: A Modal Logic for Chopping Intervals. In *Journal of Logic and Computation*, Vol. 1, pp. 453-476, Oxford University Press, 1991.

[18] Zhou Chaochen and Michel H. Hansen: An Adequate First Order Interval Logic. *UNU/IIST Report No. 91*, December 1996.

[19] Zhou Chaochen, C.A.R. Hoare and A.P. Ravn: A Calculus of Durations. In *Information Processing Letters*, Vol. 40, No. 5, pp. 269-276, 1991.