INMO–2001

February 4, 2001

1. Let ABC be a triangle in which no angle is 90°. For any point P in the plane of the triangle, let $A_1, B_1, C_1$ denote the reflections of $P$ in the sides $BC, CA, AB$ respectively. Prove the following statements:

(a) If $P$ is the incentre or an excentre of $ABC$, then $P$ is the circumcentre of $A_1B_1C_1$;
(b) If $P$ is the circumcentre of $ABC$, then $P$ is the orthocentre of $A_1B_1C_1$;
(c) If $P$ is the orthocentre of $ABC$, then $P$ is either the incentre or an excentre of $A_1B_1C_1$.

2. Show that the equation

$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$

has infinitely many solutions in integers $x, y, z$.

3. If $a, b, c$ are positive real numbers such that $abc = 1$, prove that

$$a^{b+c}b^{c+a}c^{a+b} \leq 1.$$ 

4. Given any nine integers show that it is possible to choose, from among them, four integers $a, b, c, d$ such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

5. Let $ABC$ be a triangle and $D$ be the mid-point of side $BC$. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15^\circ$. Show that $\angle ADC$ is obtuse. Further, if $O$ is the circumcentre of $ADC$, prove that triangle $AOD$ is equilateral.

6. Let $\mathcal{R}$ denote the set of real numbers. Find all functions $f : \mathcal{R} \to \mathcal{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

for all $x, y$ in $\mathcal{R}$. 