INMO–2003

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1. Consider an acute triangle ABC and let P be an interior point of ABC . Suppose the lines BP and CP , when produced , meet AC and AB in E and F respectively . Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC . Show that DK bisects $\angle EKF$.
2. Find all primes p and q ,and even numbers $n > 2$, satisfying the equation
$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$
3. Show that for every real number a the equation
$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$
has at least one non-real root and find the sum of all the non-real roots of the equation .
4. Find all 7-digit numbers formed by using only the digits 5 and 7 , and divisible by both 5 and 7 .
5. Let ABC be a triangle with sides a, b, c . Consider a triangle $A_1B_1C_1$ with sides equal to $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$. Show that
$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$
where $[XYZ]$ denotes the area of the triangle XYZ .
6. In a lottery tickets are given nine-digit numbers using only the digits $1, 2, 3$. THey are also coloured red , blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours . Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green . Determine , with proof , the colour of the ticket bearing the number 123123123 .