INMO–2005

February 6, 2005

1. Let $M$ be the midpoint of side $BC$ of a triangle $ABC$. Let the median $AM$ intersect the incircle of $ABC$ at $K$ and $L$, $K$ being nearer to $A$ than $L$. If $AK = KL = LM$, prove that the sides of triangle $ABC$ are in the ratio $5 : 10 : 13$ in some order.

2. Let $\alpha$ and $\beta$ be positive integers such that $\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}$. Find the minimum possible value of $\beta$.

3. Let $p, q, r$ be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, qx^2 + 2rx + p = 0, rx^2 + 2px + q = 0,$$

have a common root, say $\alpha$. Prove that

(a) $\alpha$ is real and negative; and

(b) the remaining third equation has non-real roots.

4. All possible 6-digit numbers, in each of which the digits occur in non-increasing order (from left to right, e.g., 877550) are written as a sequence in increasing order. Find the 2005-th number in this sequence.

5. Let $x_1$ be a given positive integer. A sequence $(x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \cdots)$ of positive integers is such that $x_n$, for $n \geq 2$, is obtained from $x_{n-1}$ by adding some nonzero digit of $x_{n-1}$. Prove that

(a) the sequence has an even number;

(b) the sequence has infinitely many even numbers.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y),$$

for all $x, y, z$ in $\mathbb{R}$. (Here $\mathbb{R}$ denotes the set of all real numbers.)