INMO–2006

1. In a nonequilateral triangle $ABC$, the sides $a, b, c$ form an arithmetic progression. Let $I$ and $O$ denote the incentre and circumcentre of the triangle respectively.

   (i) Prove that $IO$ is perpendicular to $BI$.

   (ii) Suppose $BI$ extended meets $AC$ in $K$, and $D, E$ are the midpoints of $BC, BA$ respectively. Prove that $I$ is the circumcentre of triangle $DKE$.

2. Prove that for every positive integer $n$ there exists a unique ordered pair $(a, b)$ of positive integers such that

   \[ n = \frac{1}{2}(a + b - 1)(a + b - 2) + a. \]

3. Let $X$ denote the set of all triples $(a, b, c)$ of integers. Define a function $f : X \to X$ by

   \[ f(a, b, c) = (a + b + c, ab + bc + ca, abc). \]

   Find all triples $(a, b, c)$ in $X$ such that $f(f(a, b, c)) = (a, b, c)$.

4. Some 46 squares are randomly chosen from a $9 \times 9$ chess board and are coloured red. Show that there exists a $2 \times 2$ block of 4 squares of which at least three are coloured red.

5. In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120^\circ$, and $\angle ABD = 30^\circ$. Prove that

   (i) $c \geq a + b$;

   (ii) $|\sqrt{c + a} - \sqrt{c + b}| = \sqrt{c} - a - b$.

6. (a) Prove that if $n$ is a positive integer such that $n \geq 4011^2$, then there exists an integer $l$ such that $n < l^2 < (1 + \frac{1}{2005})n$.

   (b) Find the smallest positive integer $M$ for which whenever an integer $n$ is such that $n \geq M$, there exists an integer $l$, such that $n < l^2 < (1 + \frac{1}{2005})n$. 