INMO–1990

Time : 3 hours

Attempt as many questions as you possibly can.

1. Given the equation
$$x^4 + px^3 + qx^2 + rx + s = 0$$
has four real, positive roots, prove that

   (a) $pr - 16s \geq 0$
   (b) $q^2 - 36s \geq 0$

   with equality in each case holding if and only if the four roots are equal.

2. Determine all non-negative integral pairs (x, y) for which
$$(xy - 7)^2 = x^2 + y^2.$$

3. Let f be a function defined on the set of non-negative integers and taking values in the same set. Given that

   (a) $x - f(x) = 19[x/19] - 90[f(x)/90]$ for all non-negative integers x;
   (b) $1900 < f(1990) < 2000$,

   find the possible values that $f(1990)$ can take.

   (Notation : here $[z]$ refers to largest integer that is $\leq z$, e.g. $[3.1415] = 3$).

4. Consider the collection of all three-element subsets drawn from the set $\{1, 2, 3, 4, \ldots, 299, 300\}$. Determine the number of those subsets for which the sum of the elements is a multiple of 3.

5. Let a, b, c denote the sides of a triangle. Show that the quantity
$$\frac{a}{(b+c)} + \frac{b}{(c+a)} + \frac{c}{(a+b)}$$
must lie between the limits 3/2 and 2. Can equality hold at either limits?

6. Triangle ABC is scalene with angle A having a measure greater than 90 degrees. Determine the set of points D that lie on the extended line BC, for which
$$|AD| = \sqrt{|BD||CD|}$$
where $|BD|$ refers to the (positive) distance between B and D.

7. Let ABC be an arbitrary acute angled triangle. For any point P lying within the triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, CA respectively. Determine the set of all possible positions of the point P for which the triangle DEF is isosceles. For which position of P will the triangle DEF become equilateral?