# INMO–1993

**Time : 4 hours**

**Attempt as many questions as you possibly can.**

**Use of calculating aids not permitted**

1. The diagonals $AC$ and $BD$ of a cyclic quadrilateral $ABCD$ intersect at $P$. Let $O$ be the circumcenter of triangle $APB$ and $H$ be the orthocenter of triangle $CPD$. Show that the points $H$, $P$, $O$ are collinear.
2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial in which $a$ and $b$ are integers. Given any integer $n$, show that there is an integer $M$ such that

$$P(n) \cdot P(n+1) = P(M).$$

3. If $a$, $b$, $c$, $d$ are 4 non-negative real numbers and $a + b + c + d = 1$, show that

$$ab + bc + cd \leq 1/4.$$

4. Let $ABC$ be a triangle in a plane $\Sigma$. Find the set of all points $P$ (distinct from $A$, $B$, $C$) in the plane $\Sigma$ such that the circumcircles of triangles $ABP$, $BCP$ and $CAP$ have the same radii.
5. Show that there is a natural number $n$ such that $n!$ when written in decimal notation (that is, in base 10) ends exactly in 1993 zeros.
6. Let $ABC$ be triangle right-angled at $A$ and $S$ be its circumcircle. Let $S_1$ be the circle touching the lines $AB$ and $AC$ and the circle $S$ internally. Further let $S_2$ be the circle touching the lines $AB$ and $AC$, and the circle $S$ externally. If $r_1$ and $r_2$ be the radii of the circles $S_1$ and $S_2$ respectively, show that

$$r_1 \cdot r_2 = 4(\text{ area } \Delta ABC).$$

7. Let $A = \{1, 2, 3, \ldots, 100\}$ and $B$ be a subset of $A$ having 53 elements. Show that $B$ has two distinct elements $x$ and $y$ whose sum is divisible by 11.
8. Let $f$ be a bijective (1-1 and onto) function from $A = \{1, 2, 3 \ldots, n\}$ to itself. Show that there is positive number $M \geq 1$ such that

$$f^M(i) = f(i), \text{ for each } i \text{ in } A.$$

$f^M$ denotes the composite function $\underbrace{f \circ f \circ f \circ \ldots \circ f}_{M \text{ times}}$.

9. Show that there exists a convex hexagon in the plane such that
   (a) all its interior angles are equal,
   (b) all its sides are 1, 2, 3, 4, 5, 6 in some order.