

## RMO–2000

1. Let  $AC$  be a line segment in the plane and  $B$  a point between  $A$  and  $C$ . Construct isosceles triangles  $PAB$  and  $QBC$  on one side of the segment  $AC$  such that  $\angle APB = \angle BQC = 120^\circ$  and an isosceles triangle  $RAC$  on the other side of  $AC$  such that  $\angle ARC = 120^\circ$ . Show that  $PQR$  is an equilateral triangle.
2. Solve the equation  $y^3 = x^3 + 8x^2 - 6x + 8$ , for positive integers  $x$  and  $y$ .
3. Suppose  $(x_1, x_2, \dots, x_n, \dots)$  is a sequence of positive real numbers such that  $x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \dots$ , and for all  $n$

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \dots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all  $k$  the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \dots + \frac{x_k}{k} \leq 3.$$

4. All the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000-th number in this list.
5. The internal bisector of angle  $A$  in a triangle  $ABC$  with  $AC > AB$ , meets the circumcircle  $\Gamma$  of the triangle in  $D$ . Join  $D$  to the centre  $O$  of the circle  $\Gamma$  and suppose  $DO$  meets  $AC$  in  $E$ , possibly when extended. Given that  $BE$  is perpendicular to  $AD$ , show that  $AO$  is parallel to  $BD$ .
6. (i) Consider two positive integers  $a$  and  $b$  which are such that  $a^a b^b$  is divisible by 2000. What is the least possible value of the product  $ab$ ?  
(ii) Consider two positive integers  $a$  and  $b$  which are such that  $a^b b^a$  is divisible by 2000. What is the least possible value of the product  $ab$ ?
7. Find all real values of  $a$  for which the equation  $x^4 - 2ax^2 + x + a^2 - a = 0$  has all its roots real.