

RMO–2001

1. Let BE and CF be the altitudes of an acute triangle ABC , with E on AC and F on AB . Let O be the point of intersection of BE and CF . Take any line KL through O with K on AB and L on AC . Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF . Prove that FM is parallel to EN .
2. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.
3. Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99} \right] = \left[\frac{x}{101} \right]$$

(Here $[z]$ denotes, for any real z , the largest integer not exceeding z ; e. g. $\left[\frac{7}{4} \right] = 1$)

4. Consider an $n \times n$ array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Suppose each row consists of the n numbers $1, 2, \dots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are $1, 2, \dots, n$ in some order.

5. In a triangle ABC , D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72^\circ$.
6. If x, y, z are the sides of a triangle, then prove that

$$|x^2(y - z) + y^2(z - x) + z^2(x - y)| < xyz.$$

7. Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 odd integers by a multiple of 2001.