

RMO–2003

1. Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90^\circ$. Suppose M and N are points on the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45^\circ$.
2. If n is an integer greater than 7, prove that $\binom{n}{7} - \left\lfloor \frac{n}{7} \right\rfloor$ is divisible by 7. [Here $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also for any real number x , $[x]$ denotes the greatest integer not exceeding x .]
3. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab, b - bc, c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.
4. Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:
 - (i) $x \leq y \leq z$;
 - (ii) $x + y + z \leq 100$.
5. Suppose P is an interior point of a triangle ABC such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \frac{d(B, CA)}{d(P, CA)}, \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios. [Here $d(X, YZ)$ denotes the perpendicular distance from a point X to the line YZ .]

6. Find all real numbers a for which the equation

$$x^2 + (a - 2)x + 1 = 3|x|$$

has exactly three distinct real solutions in x .

7. Consider the set $X = \{1, 2, 3, \dots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that
 - (a) $A \cup B = X$;
 - (b) $\text{prod}(A)$ is divisible by $\text{prod}(B)$, where for any finite set of numbers C , $\text{prod}(C)$ denotes the product of all numbers in C ;
 - (c) the quotient $\text{prod}(A)/\text{prod}(B)$ is as small as possible.