Pattern Recognition Letters 11 (1990) 197–206
North-Holland

March 1990

Fuzzy thresholding: mathematical framework, bound functions and weighted moving average technique

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Received 28 April 1989

Abstract: The problem of histogram sharpening and thresholding by minimising greylevel fuzziness is considered. The earlier work on the said problem consists only of algorithms without mathematical justification of the findings. For example, the choices of appropriate membership function and the optimum value of its window size (band width) for detecting thresholds were made experimentally with iterative manner.

The present work provides a complete theoretical formulation of the same and establishes the criteria regarding the choices of membership function and its window size (band width). The variation in membership function is seen to be restricted by bound functions, thus enabling the method of segmentation more flexible but effective. Finally, the method can be viewed as a weighted moving average technique, greyness ambiguity being the weights.

Key words: Image thresholding, greyness ambiguity, fuzzy sets, bound functions.

1. Introduction

A measure of ambiguity (fuzziness) in grey level of an image is seen to be provided [1] by the terms index of fuzziness [2], entropy [3] and index of non-fuzziness [4]. Since these terms basically reflect the measure of closeness of greytone image to its two-tone version, they provide a quantitative measure of image ambiguity [1] when the cross-over point is set to a predetermined value. Modification of cross-over point will result in variation in these values and so a set of minima may be obtained corresponding to the optimum threshold levels of the image.

The above concept was used earlier experimentally by Pal et al. [5] and Pal and Rosenfeld [6] to detect thresholds for various bimodal and multimodal images. They considered only Zadeh's standard S function [7] over an interval of length c (window size) in extracting a fuzzy subset 'bright image' from the image. But the authors did not provide any mathematical basis of either the choice of membership function or their findings. For example, the limitation of using any other type of membership function is neither theoretically nor even experimentally justified. The observations on the choice of c (which is critical for detecting valleys) was provided only experimentally. No mathematical reason was given on the choice of optimal value of c in order to detect a valley.

The present work provides a mathematical formulation of the aforesaid method, establishes theoretically the choice of optimum c and the selection of membership function and justifies the said experimental results. The frame work takes into consideration all possible membership functions and histograms. The relation between c and the length of the interval between two peaks of histogram is established. The effect of variation of membership function (i.e., the limitation on the choice of g) on the results is mathematically described. It is then found out that the requisite membership function g may be confined within the bounds of Murthy and Pal [8] and it possesses symmetry in ambiguity around the cross-over point. With this, the method is therefore seen to be flexible enough in selecting its input membership function keeping the output satisfactory.

The present investigation can not only be regarded as a completion of the earlier work, but also be viewed as a generalization of the same. Finally, the method has been shown to be equivalent to a weighted moving average technique [9].

2. Greyness ambiguity and threshold selection

Let X be an image of $L+1$ levels, M rows and N columns and μ be a membership function defined on levels. Let $\mu(x_{mn})$ denote the grade of possessing some property (e.g. brightness) by the (m,n) pixel of intensity x_{mn}, $m=1,\ldots,M$ and $n=1,\ldots,N$. The index of fuzziness ($\gamma(X)$), entropy ($E(X)$) and index of nonfuzziness ($\eta(X)$) are defined below [1]:

$$\gamma(X) = \frac{2}{MN}\sum_m\sum_n \min[\mu(x_{mn}),(1-\mu(x_{mn})]$$
$$= \frac{2}{MN}\sum_m\sum_n |\mu(x_{mn}) - \mu_{\underset{\sim}{X}}(x_{mn})| \qquad (1a)$$

where $\mu_{\underset{\sim}{X}}$ represents the nearest two-tone version of μ.

$$E(X) = \frac{1}{MN \ln 2}\sum_m\sum_n \mathrm{Sn}(\mu(x_{mn})) \qquad (1b)$$

with Shannon's function

$$\mathrm{Sn}(\mu(x_{mn})) = -\mu(x_{mn})\log\mu(x_{mn}) - (1-\mu(x_{mn}))\log(1-\mu(x_{mn})). \qquad (1c)$$

$$\eta(X) = \frac{1}{MN}\sum_m\sum_n [1 - |2\mu(x_{mn}) - 1|]. \qquad (1d)$$

Intuitively ambiguity in greyness should be maximum when $\mu(x_{mn}) = 0.5$ and it should decrease as $\mu(x_{mn})$ moves away from 0.5. The above mentioned measures possess this property and hence they can be considered to represent the greyness ambiguity in X.

Observe that $\eta(X)$ is the same as $\gamma(X)$ because of the reason mentioned below.

(a) Let $\mu(x_{mn}) \leqslant 1/2$. Then

$$1 - |2\mu(x_{mn}) - 1| = 2\mu(x_{mn}).$$

(b) Let $\mu(x_{mn}) \geqslant 1/2$. Then

$$1 - |2(\mu(x_{mn})) - 1| = 2(1-\mu(x_{mn})).$$

Let $f(l)$ denote the number of occurrences of the level l. Equations (1) can then be written as

$$\gamma(X) = \frac{2}{MN}\sum T(l)f(l) \qquad (2a)$$

$$\text{with } T(l) = \min[\mu(l), 1-\mu(l)], \qquad (2b)$$

$$E(X) = \frac{1}{MN \ln 2}\sum \mathrm{Sn}(\mu(l))f(l). \qquad (3)$$

The concept of using index of fuzziness for threshold selection is described below. Similar argument holds for entropy also. Let Zadeh's standard S function [7] be considered μ here.

$$\begin{aligned}\mu(x_{mn}; p,q,r) &= 0 \quad \text{if } x_{mn} \leqslant p,\\ &= 2[(x_{mn}-p)/(r-p)]^2 \quad \text{if } p \leqslant x_{mn} \leqslant q,\\ &= 1 - 2[(x_{mn}-r)/(r-p)]^2 \quad \text{if } q \leqslant x_{mn} \leqslant r,\\ &= 1 \quad \text{if } x_{mn} \geqslant r \end{aligned} \qquad (4)$$

with $q = \frac{1}{2}(p+r)$ and $\Delta q = r - q = q - p$. The parameter q is the cross-over point. The window length $= r - p = 2\,\Delta q$.

Let us, for example, consider the object and background segmentation of a bimodal histogram. Now, for an image X, the fuzzy measures basically compute the distance between its brightness property μ_X and its nearest two-tone property $\mu_{\underset{\sim}{X}}$. Since, $\underset{\sim}{X}$ is dependent of the cross-over point q, a proper selection of q (and hence the membership function) may therefore be obtained which will result in minimum value of these measures γ and E. This minimum value corresponds to appropriate segmentation of the image and q may be taken as optimum threshold. This is optimum in the sense that, for any other choice of q, the γ or E measure will be greater than this.

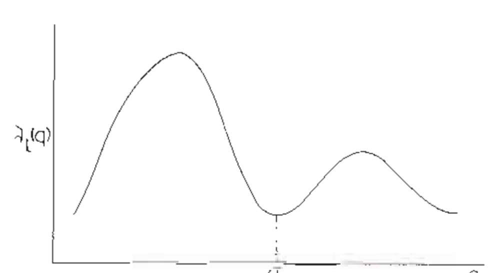

Figure 1 Graph showing index of fuzziness values vs. cross-over points. q_0 is optimal.

The corresponding $\mu(x_{mn})$ plane can be regarded as a fuzzy segemented version of X. For obtaining its non-fuzzy (crisp) version, the cross-over point q (having maximum ambiguity) was considered above as the threshold between object and background. The above concept can similarly be extended to a multimodal image where there would be several minima corresponding to different valley points of the histogram.

Algorithm for greylevel thresholding

The fuzzy membership function $\mu(x_{mn}; p, q, r)$ (equation (4)) is considered. The function μ is shifted over the interval $[0, L]$ by varying p, q and r but keeping Δq fixed. When Δq is fixed, the whole function μ can be determined uniquely given q. Indices of fuzziness are calculated for every μ, i.e., for every q. The valley points of $\gamma(q)$ are taken to be the detected thresholds (unambiguous valley points) of the histogram of the input image. The algorithm is thus seen to be able to sharpen an input histogram by removing the local variations and ambiguities in the vicinity of its valleys.

In this algorithm $c = 2\,\Delta q$ is the length of the interval which is shifted over the entire dynamic range. As c decreases, the $\mu(x_{mn})$ plane would have more intensified contrast around the cross-over point resulting in decrease of ambiguity in X. As a result the possibility of detecting some undesirable thresholds (spurious minima) increases because of the smaller value of Δq.

On the other hand, increase of c results in a higher value of fuzziness and thus leads towards the possibility of losing some of the weak minima.

Though the earlier works [5,6] used this concept, the mathematical formulation of the problem was not provided. For example, it was reported that if c is greater than the distance between the modes, then the corresponding valley point may be lost [5]. But the mathematical justification of this finding was not given. Similar is the case with the selection of the membership function where only the function shown in equation (4) was considered. The consequences of using other types of membership functions are neither mathematically nor experimentally studied.

In Section 3, a mathematical formulation of the problem and some of its consequences are stated. The relation between c (window size) and the distance between modes is also established in Section 3. In Section 4, different types of membership functions are taken and the corresponding changes on the thresholds of histogram are discussed. Section 5 describes the relation between the moving average method [9] and the method presented here.

3. Mathematical formulation of histogram thresholding

We shall assume continuous functions for the formulation and proofs. Similar results hold in discrete cases also. The histogram will be represented by f, the membership function by g and the index of fuzziness by $M_0 H$ where M_0 is a constant ($M_0 = 2/MN$ of equation (2a)) and H represents the other part of equation (2a). (The summation sign should be changed to an integral because of continuity.)

Theorem 1. *Let f: $[0, L] \to [0, \infty)$ be such that*

(i) *f is continuous,*

(ii) *$f(a) = f(b)$, $a < b$ and f has local maximums at a and b,*

(iii) *$y_0 = \frac{1}{2}(a + b)$, f has a local minimum at y_0,*

(iv) *f is symmetric around y_0 in the interval $[a, b]$, and*

(v) *f is convex in $[a, b]$.*

Let g: $[0, c] \to [0, 1]$ be such that

(i) *g is continuous, $g(0) = 0$, $g(c) = 1$,*

(ii) *g is monotonically non-decreasing, and*

(iii) *$g(x) = 1 - g(c - x)$ $\forall x \in [0, c]$* (5)

where $c > 0$ is the length of the window.

Let $c < b - a$. Let $\delta = \frac{1}{2}(b - a - c)$. Let

$$H_g(y) = \int_0^{c/2} g(x) f(y - c/2 + x)\,\mathrm{d}x + \int_{c/2}^{c} (1 - g(x)) f(y - c/2 + x)\,\mathrm{d}x.$$

(Observe that $M_0 H_g(y)$ gives the index of fuzziness in the interval $(y - c/2, y + c/2)$.) Then

$$H_g(y) \geqslant H_g(y_0) \quad \forall y \in (y_0 - \delta, y_0 + \delta).$$

Proof. Let $0 < \varepsilon < \delta$. Let $y = y_0 - \varepsilon$. We shall show that $H_g(y) \geqslant H_g(y_0)$.

$$H_g(y) = \int_0^{c/2} g(x) f(y_0 - \varepsilon - c/2 + x)\,dx + \int_{c/2}^{c} (1 - g(x)) f(y_0 - c/2 - \varepsilon + x)\,dx = I_1 + I_2 \qquad \text{(say)}.$$

We shall simplify I_2 now with the transformation $x = c - z$

$$I_2 = -\int_{c/2}^{0} [1 - g(c - z)] \times f(y_0 - c/2 - \varepsilon + c - z)\,dz = \int_0^{c/2} g(z) f(y_0 + c/2 - \varepsilon - z)\,dz = \int_0^{c/2} g(z) f(y_0 - c/2 + \varepsilon + z)\,dz.$$

(Thus f is symmetric in $[a, b]$.) So

$$H_g(y) = \int_0^{c/2} g(x)\,[f(y_0 - \varepsilon - c/2 + x) + f(y_0 - c/2 + \varepsilon + x)]\,dx$$
$$= 2\int_0^{c/2} g(x)\,[\tfrac{1}{2} f(y_0 - \varepsilon - c/2 + x) + \tfrac{1}{2} f(y_0 + \varepsilon - c/2 + x)]\,dx$$
$$\geqslant 2\int_0^{c/2} g(x) f\{\tfrac{1}{2}[y_0 - \varepsilon - c/2 + x + y_0 + \varepsilon - c/2 + x]\}\,dx$$
$$= 2\int_0^{c/2} g(x) f(y_0 - c/2 + x)\,dx. \qquad (6)$$

Now

$$H_g(y_0) = \int_0^{c/2} g(x) f(y_0 - c/2 + x)\,dx + \int_{c/2}^{c} (1 - g(x)) f(y_0 - c/2 + x)\,dx = J_1 + J_2 \qquad \text{(say)}.$$

By applying the same calculation of I_2 to J_2 it can be shown that

$$J_2 = \int_0^{c/2} g(x) f(y_0 - c/2 + x)\,dx$$

i.e.

$$H_g(y_0) = 2\int_0^{c/2} g(x) f(y_0 - c/2 + x)\,dx \leqslant H_g(y) \qquad \text{(from (6))}.$$

For $y = y_0 + \varepsilon$ when $0 < \varepsilon < \delta$, a similar proof holds. Hence the theorem. □

Remark 1. (a) A similar proof can be given if the entropy is taken to be the grey level ambiguity measure.

(b) δ in the above theorem will give an idea of the length of the interval in which $H_g(y) \geqslant H_g(y_0)$. For c being close to $b - a$, δ will be very small and the valley will be obtained in a smaller interval. In practical cases the valley may become invisible also. The case of $c = b - a$ is tackled below in Note 1.

Note 1. Let f and g satisfy the same assumptions as in Theorem 1. Let $\varepsilon > 0$ be a small quantity, $c = b - a$ and $y = y_0 - \varepsilon$. So

$$H_g(y) = \int_0^{\varepsilon} g(x) f(y_0 - \varepsilon - c/2 + x)\,dx + \int_{\varepsilon}^{c/2} g(x) f(y_0 - \varepsilon - c/2 + x)\,dx + \int_{c/2}^{c-\varepsilon} (1 - g(x)) f(y_0 - \varepsilon - c/2 + x)\,dx$$

$$+ \int_{c-\varepsilon}^{c} (1 - g(x)) f(y_0 - \varepsilon - c/2 + x)\,\mathrm{d}x$$

$$= I_1 + I_2 + I_3 + I_4 \quad \text{(say)}.$$

$$H_g(y_0) = \int_0^{\varepsilon} g(x) f(y_0 - c/2 + x)\,\mathrm{d}x$$

$$+ \int_{\varepsilon}^{c/2} g(x) f(y_0 - c/2 + x)\,\mathrm{d}x$$

$$+ \int_{c/2}^{c-\varepsilon} (1 - g(x)) f(y_0 - c/2 + x)\,\mathrm{d}x$$

$$+ \int_{c-\varepsilon}^{c} (1 - g(x)) f(y_0 - c/2 + x)\,\mathrm{d}x$$

$$= J_1 + J_2 + J_3 + J_4 \quad \text{(say)}.$$

Now, as in Theorem 1, I_3 can be proved to be equal to

$$\int_{\varepsilon}^{c/2} g(x) f(y_0 + x + \varepsilon - c/2)\,\mathrm{d}x.$$

Similar to Theorem 1, $I_2 + I_3$ can be proved to be greater than or equal to

$$2 \int_{\varepsilon}^{c/2} g(x) f(y_0 - c/2 + x)\,\mathrm{d}x = J_2 + J_3.$$

Now $J_1 + J_4$ can be proved to be equal to

$$2 \int_0^{\varepsilon} g(x) f(y_0 - c/2 + x)\,\mathrm{d}x.$$

I_4 can be shown to be equal to

$$\int_0^{\varepsilon} g(x) f(y_0 - c/2 + \varepsilon + x)\,\mathrm{d}x.$$

Therefore,

$$I_1 + I_4 - J_1 - J_4$$

$$= \int_0^{\varepsilon} g(x)\,[f(y_0 - \varepsilon - c/2 + x)$$

$$+ f(y_0 - c/2 + x + \varepsilon) - 2f(y_0 - c/2 + x)]\,\mathrm{d}x.$$

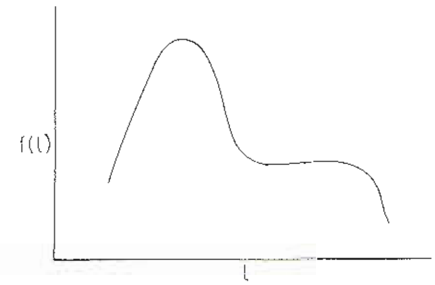

Figure 2. Histogram in which a mode can not be precisely denoted.

If ε is sufficiently small, this difference may (not always) become negligible because g is continuous and f is continuous which in turn gives $H_g(y_0) \leqslant H_g(y)$. So for $c = b - a$, practically, it is not always guaranteed that $H_g(y_0) < H_g(y)$. For $c > b - a$, a similar conclusion can be arrived at. So it can be conclusively stated that, for achieving a valley in $H_g(y)$ corresponding to a valley in f, the window length c should be less than the distance between two peaks, if f and g satisfy the conditions of Theorem 1.

In practical problems, the modes of histograms may not be found out exactly. For example, in Figure 2, though it appears that there are two modes, the value of the second mode is not exactly known. In Figure 3, another histogram is shown, where the two modes are known more or less accurately but the convexity and symmetry properties do not hold and also there are many other local minima. By using fuzzy membership functions for sharpening the histogram, we would like to remove the redundant local minima of the histogram of Figure 3, so that $H_g(y)$ in Theorem 1 would have one minimum in between the two modes. That is, the membership function should be taken in such a way that $H_g(y)$

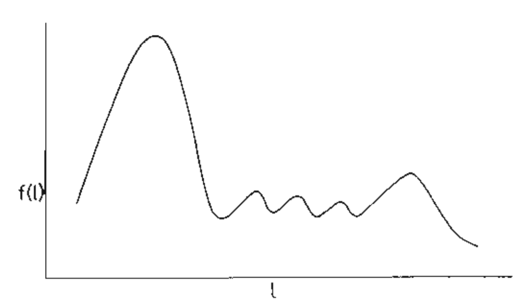

Figure 3. Histogram where convexity and symmetry properties are not satisfied.

should remove 'unnecessary' local minima of the histogram. We shall show below that if the value of c is very small then $H_g(y)$ would give many local minima (Example 1).

Example 1. Consider Figure 4 where a histogram f is shown. Though the prominent modes are a_1 and a_2, there are other local maxima namely a_3, a_4, a_5 and a_7. Let us consider a_1 and a_5. y_0 is the only local minimum between a_1 and a_5. f is convex in the interval a_1 and a_5. Let a_6 be such that $f(a_6) = f(a_5)$ and $a_1 < a_6 < y_0 < a_5$. Now suppose that f is symmetric in the interval (a_6, a_5) around y_0. Let $c < a_5 - a_6$. Then any g satisfying the conditions of Theorem 1 would have y_0 as a local minimum. But the detection of y_0 as a threshold is desirable or not depends on the problem. For the same histogram shown in Figure 4 and for a two-class problem $H_g(y)$ may produce other local minima if the value of c is not big enough. In practical problems, whether the conditions of convexity and symmetry are satisfied or not, it is better to take c to be $\geqslant a_5 - a_6$ if the detection of y_0 is to be avoided.

From Example 1, it is apparent that the value of c cannot be very small compared to the difference between the modes. In this section, though all the results are stated for the index of fuzziness, the same conclusions would hold for the entropy also.

In the next section the effect of various types of membership functions on the valley points are observed.

4. Various membership functions and greyness ambiguity

In the previous section the relation between c and the difference in local minima is established. In this section, different types of membership functions are examined for the threshold selection using greyness ambiguity. In Example 2, the same histogram of Example 1 is considered to show that some types of membership functions may provide undesirable results.

Example 2. The histogram under consideration is the one shown in Figure 4. Consider $g_{\varepsilon,\delta}$ as shown in Figure 5 for $\varepsilon > 0$ and $\delta > 0$. (Though a specific form of g is presented below, any form with the same idea would suffice the purpose.)

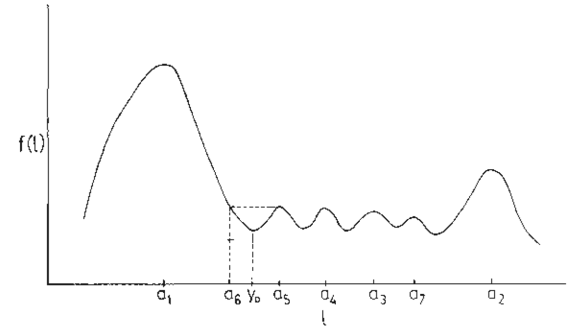

Figure 4. Histogram of Examples 1 and 2.

$$g_{\varepsilon,\delta}(x) = \frac{2\varepsilon x}{c - a_5 + a_6 + \delta} \quad \text{for } 0 \leqslant x \leqslant \tfrac{1}{2}(c - a_5 + a_6 + \delta) = A_1,$$

$$= \frac{1}{2} + \frac{(1 - 2\varepsilon)(x - \frac{1}{2}c)}{a_5 - a_6 - \delta} \quad \text{for } A_1 \leqslant x \leqslant \tfrac{1}{2}c,$$

$$= 1 - g_{\varepsilon,\delta}(c - x) \quad \text{for } \tfrac{1}{2}c \leqslant x \leqslant c$$

where $0 < \varepsilon$ is a small quantity, $0 < \delta < a_5 - a_6$, a_5 and a_6 are as defined in Example 1, c is any positive number and $A_2 = c - A_1$. The essential differences between g of Example 1 and $g_{\varepsilon,\delta}$ of this example are listed in Table 1.

Note 2. In order that y_0 of Figure 4 should not be detected as a valley point of the greyness ambiguity function, not only the value of $c > a_5 - a_6$, but also the membership function $g_{\varepsilon,\delta}$ of Example 2 is to be avoided. $g_{\varepsilon,\delta}$ has most of its variation concentrated

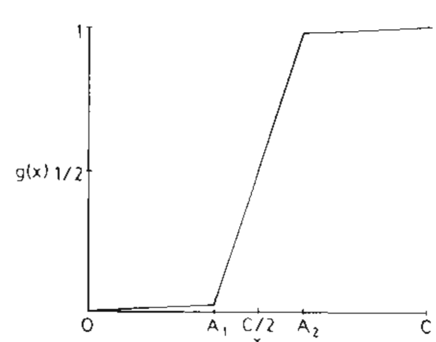

Figure 5. Membership function of Example 2.

Table 1

	g of Example 1	$g_{\varepsilon,\delta}$ of Example 2
1.	The length of domain of g is less than $a_5 - a_6$.	The length of domain of $g_{\varepsilon,\delta}$ is greater than $a_5 - a_6$.
2.	Outside the interval of length $a_5 - a_6$, g takes values '0' or '1' so that greyness ambiguity will be zero for those valnes.	Outside the middle interval of length $a_5 - a_6 - \delta$, either Shannon's function or $\min(g_{\varepsilon,\delta}(x), 1 - g_{\varepsilon,\delta}(x))$ takes very small values (because ε can be made arbitrarily small) so that after the multiplication with f, the result would be insignificant. That is, $g_{\varepsilon,\delta}$ serves the same pnrpose as g of Example 1. Hence valley y_0 will be detected.

in a small interval in the middle of $[0,c]$ and has little variation in the rest. So this function would not satisfy the bounds of Murthy and Pal [8]. (The bounds of Murthy and Pal are described in the Appendix.) In the practical problems where assumptions of convexity and symmetry are not satisfied for f, it is imperative that the functions of the sort $g_{\varepsilon,\delta}$ are to be avoided.

We will now show that if most of the variation in g is concentrated towards one of the end points of the interval $[0,c]$, it is inadvisable to consider that function (Note 3). The argument will be similar to that of Example 2.

Note 3. Let g be a function (Figure 6) from $[0,c]$ to $[0,1]$ such that

(i) $g(0) = 0$, $g(c) = 1$, g is monotonically non-decreasing,

(ii) there exists a point $z \in [3c/4, c)$ such that $g(z) = \frac{1}{2}$, and

(iii) there exists z_0, $c/2 \leqslant z_0 < z$ such that $g(z_0) < \varepsilon$ where ε is a small positive value.

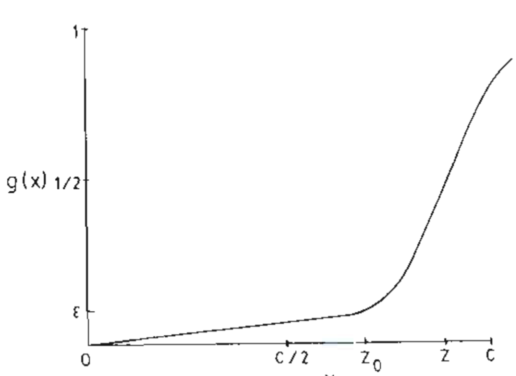

Figure 6. Membership function of Note 3.

That is, most of the variation in g is concentrated towards the end point c. The multiplication of the heights of the histogram with either Shannon's function or the index of fuzziness would be insignificant in the interval $[0,c/2]$ if ε is taken suitably. This would essentially result in a membership function g whose domain is of length $c/2$ but not c.

That means, once the value of c is chosen, the variation of g should not be concentrated mostly on a small interval towards the end point of the interval $[0,c]$. Similarly, it can be argued that it cannot be concentrated towards the starting point of the interval $[0,c]$. From Note 2, it is apparent that the variation in g cannot be concentrated in a small interval in the middle of $[0,c]$.

Observe that, if the variation in g is concentrated towards one of the end points of g, or in a small interval in the middle of $[0,c]$, then it can not satisfy the bounds of Murthy and Pal [8]. The conclusion is that, g can be taken to be a function satisfying the above bounds. In Figure 7, such bound functions are shown. Once g satisfies the bounds, then the form of g may be taken as

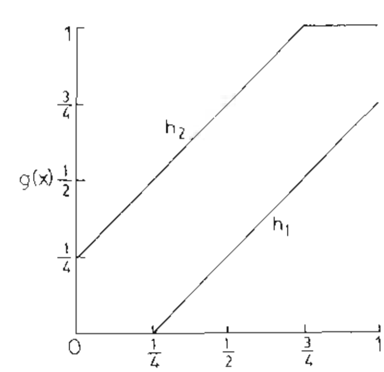

Figure 7. Bound functions.

$$g(x) = 1 - g(c - x) \quad \forall x \in [0, c],$$

because this would be able to detect the valley, in case it is present.

5. Method of moving averages

The method of moving averages is a standard technique in statistics [9] for smoothening the histogram. The method is described below, and its relation with histogram smoothing by minimising the greyness ambiguity is described.

Moving averages

The method of moving averages for window size 3 is described below. The data is the output of a certain factory for every month in a year. See Table 2. Observe that for a continuous histogram f, window size c, the moving average method would give

$$\left(\int_0^c f(y - c/2 + x)\,\mathrm{d}x\right)/c$$

for every point y.

A generalization to the moving average method is to give unequal weights to the frequencies, i.e. for weight function h, the transformed frequency at the point y is

$$\frac{\int_0^c h(x)\, f(y - c/2 + x)\,\mathrm{d}x}{\int_0^c h(x)\,\mathrm{d}x}.$$

Now, in order to find the local minimum value of the transformed frequencies it is sufficient to deal with

$$\int_0^c h(x)\, f(y - c/2 + x)\,\mathrm{d}x$$

since $\int_0^c h(x)\,\mathrm{d}x$ is a constant.

While selecting thresholds for histogram segmentation, we minimised

$$\int_0^c h(x)\, f(y - c/2 + x)\,\mathrm{d}x$$

where $h(x)$ depends on the greyness ambiguity measure. So the methods described in this paper are nothing but a generalization of the technique of moving averages and segmenting on the basis of the transformed histogram. In other words, the method may be said to be a weighted moving average threshold selection method, where the weights are the ambiguity values.

Table 2

Month	Output	Result of moving average method for window size 3
January	a_1	
February	a_2	$(a_1 + a_2 + a_3)/3$
March	a_3	$(a_2 + a_3 + a_4)/3$
April	a_4	$(a_3 + a_4 + a_5)/3$
May	a_5	$(a_4 + a_5 + a_6)/3$
June	a_6	$(a_5 + a_6 + a_7)/3$
July	a_7	$(a_6 + a_7 + a_8)/3$
August	a_8	$(a_7 + a_8 + a_9)/3$
September	a_9	$(a_8 + a_9 + a_{10})/3$
October	a_{10}	$(a_9 + a_{10} + a_{11})/3$
November	a_{11}	$(a_{10} + a_{11} + a_{12})/3$
December	a_{12}	–

6. Conclusions

Minimising the greyness ambiguity by different fuzzy measures has been proved to be a useful tool for formulating a method of segmentation or sharpening of a grey tone image. Any kind of monotonically nondecreasing membership function satisfying the bounds and equation (5) can be used for the above mentioned purposes. This, in turn, makes the approach flexible.

In the earlier reports [5,6], Zadeh's S function was used as a membership function. Note that this function satisfies the bounds and equation (5). Therefore, the theory described here can be viewed as a generalization of earlier works. Note also that some observations were made [5,6] on the effect of window size without any mathematical basis. The present framework provides a theoretical justification to those findings.

In addition, the present investigation visualises the problem in a generalised set up and provides flexible choices for window sizes and membership functions with a proper mathematical basis. In this sense, this can be considered as a completion of the earlier research. Furthermore, the algorithm is found to be analogous to the moving average technique.

Acknowledgements

The authors gratefully acknowledge Prof. D. Dutta Majumder for his advices and discussions during the course of work. They are also grateful to Mr. S. Chakraborty for drawing diagrams and Mrs. S. De Bhowmick for typing the manuscript.

Appendix

Bounds for membership functions

The membership function g considered throughout this paper has the following properties:

(i) g: $[0,c] \to [0,1]$ is continuous,

(ii) $g(0)=0$, $g(1)=1$ and g is monotonic.

Recently Murthy and Pal [8] formulated bounds for membership functions of the above sort in order to discard the membership functional forms which are to be avoided while representing a fuzzy set in practice. Significance of these bounds in image segmentation and analysis problems was also found to be justified [8].

The expression for bound functions is based on properties of correlation [10] between two membership functions $\delta_1(x)$ and $\delta_2(x)$. The main properties on which correlation was formulated are

P_1: If for higher values of δ_1, δ_2 takes higher values and for lower values of δ_1, δ_2 also takes lower values then $c_{\delta_1,\delta_2} > 0$ (c represents correlation).

P_2: If $\delta_1\uparrow$ and $\delta_2\uparrow$ then $c_{\delta_1,\delta_2} > 0$.

P_3: If $\delta_1\uparrow$ and $\delta_2\downarrow$ then $c_{\delta_1,\delta_2} < 0$.

($\uparrow$ denotes increases and $\downarrow$ denotes decreases.)

It is to be mentioned that P_2 and P_3 should not be considered in isolation of P_1. Had this been the case, one can cite several examples when $\delta_1\uparrow$ and $\delta_2\uparrow$ but $c_{\delta_1,\delta_2} < 0$ and $\delta_1\uparrow$ and $\delta_2\downarrow$ but $c_{\delta_1,\delta_2} > 0$. Subsequently, the types of membership functions which should not be considered in fuzzy set theory are categorised with the help of correlation. Bound functions h_1 and h_2 are accordingly derived [8]. They are

$$\begin{aligned} h_1(x) &= 0, & 0 \leqslant x \leqslant \varepsilon, \\ &= x-\varepsilon, & \varepsilon \leqslant x \leqslant 1, \\ h_2(x) &= x+\varepsilon, & 0 \leqslant x \leqslant 1-\varepsilon, \\ &= 1, & 1-\varepsilon \leqslant x \leqslant 1 \end{aligned}$$

where $\varepsilon = 0.25$.

The bounds for the membership function g considered throughout this paper are $h_1(x) \leqslant g(x) \leqslant h_2(x)$ for $x \in [0,1]$.

For x belonging to any arbitrary interval, the bound functions will be changed proportionately. For $h_1 \leqslant g \leqslant h_2$, $c_{h_1,h_2} \geqslant 0$, $c_{h_1,g} \geqslant 0$ and $c_{h_2,g} \geqslant 0$. The function g lying in between h_1 and h_2 does not have most of its variation concentrated (i) in a very small interval, (ii) towards one of the end points of the interval under consideration and (iii) towards both the end points of the interval under consideration.

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