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A fuzzy medial axis transformation based on fuzzy disks

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Abstract

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A fuzzy disk with center P is a fuzzy set in which membership depends only on distance from P. For any fuzzy set f, there is a maximal fuzzy disk $g_P^f \leqslant f$ centered at every point P, and f is the sup of the g_P^f's. (Moreover, if f is fuzzy convex, so is every g_P^f, but not conversely.) We call a set S_f of points f-sufficient if every $g_P^f \leqslant g_Q^f$ for some Q in S_f; evidently f is then the sup of the g_Q^f's. In particular, in a digital image, the set of Q's at which g^f is a (nonstrict) local maximum is f-sufficient. This set is called the *fuzzy medial axis* of f, and the set of g_Q^f's is called the fuzzy medial axis transformation (FMAT) of f. These definitions evidently reduce to the standard one if f is a crisp set. Unfortunately, for an arbitrary f, specifying the FMAT may require more storage space than specifying f itself.

Keywords. Fuzzy disk, fuzzy medial axis, fuzzy MAT.

1. Introduction

Let S be a subset of a metric space; then S is the union of the maximal disks (defined with respect to the given metric) centered at the points of S. More precisely, let P be any point of S, and let D_P^S be the maximal disk centered at P and contained in S; then S is the union of the D_P^S's. Let S' be any subset of S such that, for all $P \in S$, there exists $Q \in S'$ such that $D_P^S \subseteq D_Q^S$; we call S' a sufficient subset of S. Evidently, for any such S', S is the union of the D_Q^S's.

In particular, let S be a set of nodes in a graph, and regard the graph as a metric space under the metric defined by path length. Let S^* be the set of nodes Q of S at which D_Q^S is a local maximum (i.e. for any neighbor P of Q, $D_P^S \subseteq D_Q^S$). Then S^* is a sufficient subset of S; it is called the *medial axis* of S, and the set of D_Q^S's for $Q \in S^*$ is called the

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medial axis transformation (MAT) of S. When the graph is the set of pixels in a digital image under any of the standard definitions of 'neighbor', this reduces to a standard definition of the MAT [1].

In this paper we generalize this definition of the MAT to fuzzy subsets of a metric space. Our definition is based on the concept of a fuzzy disk, which is a fuzzy set in which membership depends only on distance from a given point [2]. Basically, the fuzzy MAT (FMAT) of a fuzzy set f is a set of fuzzy disks whose sup is f; it will be defined more precisely in Section 2; and examples will be given in Section 3. Unfortunately, it turns out that specifying the FMAT sometimes requires more storage space than specifying f itself.

2. The FMAT

There have been several generalizations of the MAT to grayscale images, e.g. using a 'gray-weighted' definition of distance [3,4], or using the methods of gray-weighted 'mathematical morphology' [5,6]; a related idea is the concept of gray-weighted thining or skeletonization [7,8]. If the gray levels are scaled to lie in the range [0, 1], we can regard the gray level of a pixel as its degree of membership in the set of high-valued ('bright') pixels; thus a grayscale image can be regarded as a fuzzy set.

The general definition of the MAT given iu Section 1 generalizes straightforwardly to fuzzy subsets of a metric space. We recall [2] that, for any metric, a *fuzzy disk* centered at the point P is a fuzzy set in which membership depends only on distance from P.

Let D be a metric space with metric d and let f be a fuzzy subset of D. For each $P \in D$, let g_P^f be the fuzzy set defined by

$$g_P^f(Q) \equiv \inf_{d(P,R)=d(P,Q)} f(R).$$

Evidently g_P^f is a fuzzy disk and $g_P^f \leqslant f$; in fact, g_P^f is the maximal fuzzy disk centered at P and not exceeding f. Moreover, it is easy to see that

$$\sup_{P \in D} g_P^f = f.$$

(*Proof.* Let $h_P^f = f(P)$ at P, $=0$ otherwise; then $h_P^f \leqslant g_P^f$, and evidently $\sup_P h_P^f = f$.)

Let D' be any subset of D such that

$$(\forall P \in D)(\exists Q \in D')[g_P^f \leqslant g_Q^f];$$

we call such a D' an f-sufficient subset of D. Evidently, for any such D' we have

$$\sup_{P \in D'} g_P^f = f.$$

In particular, let D be the set of nodes of a graph under the path metric, as in Section 1. We say that $P \in D$ is a (nonstrict) local maximum of f if P has no neighbor Q such that $g_P^f < g_Q^f$. Evidently, the set D_f of such local maxima of f is an f-sufficient subset of D, so that f is the sup of the g_P^f's for all $P \in D_f$. We call D_f the *fuzzy medial axis* of f, and we call $\{g_P^f \mid P \in D_f\}$ the *fuzzy medial axis transformation* of f. It is easily seen that if f is a crisp subset of D, these definitions reduce to those in Section 1.

It should be pointed out that although crisp disks are convex, fuzzy disks are not necessarily fuzzy convex. However, we can show that if D is the plane under the Euclidean metric, and f is fuzzy convex, then the g_P^f's are all fuzzy convex. We recall that a fuzzy subset f of the plane is called *fuzzy convex* if for all collinear triples of points P, Q, R, with Q between P and R, we have

$$f(Q) \geqslant \min[f(P), f(R)].$$

Evidently, a fuzzy disk centered at P in the Euclidean plane is fuzzy convex if and only if its membership function is a monotonically nonincreasing function of distance to P.

Proposition 1. *If f is fuzzy convex, so is every g_P^f.*

Proof. Suppose we had $g_P^f(s) > g_P^f(r)$ for some $r < s$; let $g_P^f(s) - g_P^f(r) = \varepsilon$. Let Q be a point at distance r from P such that

$$f(Q) - g_P^f(r) < \varepsilon.$$

[Since $g_P^f(r) \equiv \inf_{d(P,R)=r} f(R)$, we can find a point Q at distance r from P whose f value is equal to or arbitrarily close to the inf.] Thus

$$f(Q) < \varepsilon + g_P^f(r) = g_P^f(s).$$

Let Q_1, Q_2 be any two points at distance s from P

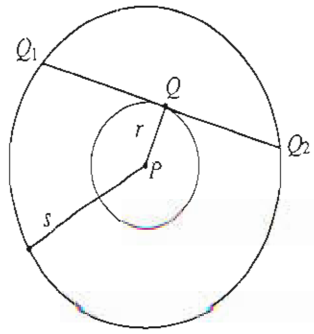

Figure 1. Illustrating the proof of Proposition 1.

such that the line segment Q_1Q_2 passes through Q, e.g. as shown in Figure 1. Then

$$f(Q_1), f(Q_2) \geqslant \inf_{d(P,R)=s} f(R) \equiv g_P^f(s),$$

while $f(Q) < g_P^f(s)$, so that the collinear triple (Q_1, Q, Q_2) violates the fuzzy convexity of f. □

The converse of Proposition 1 is false. If $f=1$ at two points and $f=0$ elsewhere, then every g_P^f is fuzzy convex, but f is not—in fact, it is not even fuzzy connected.

Since crisp disks are convex, we also have a generalization of the crisp MAT if we define the FMAT using the maximal *convex* fuzzy disk centered at (every) P and not exceeding f. We shall consider both the general definition and this 'convex' definition in our examples.

3. Examples

Let D be the digital plane (i.e., the integer-coordinate lattice points) under the chessboard metric; a 'disk' in this metric is an upright square of odd side length. Let f be zero except on an $n \times n$ array of lattice points (so that f represents an $n \times n$ digital image with gray levels in the range [0, 1]). For any pixel P, we have $g_P^f(r) = 0$ for any $r \geqslant$ the chessboard distance d_P from P to the border of the image. Thus to specify g_P^f, we need only list its values for the (integer) chessboard distances up to d_P. In an $n \times n$ digital image, the average number of values will be on the order of n, and if the number of pixels that belong to the fuzzy medial axis is also $O(n)$, the total number of values needed to specify the FMAT is $O(n^2)$.

As a toy example, consider the 5×5 digital image f shown in Figure 2a. For the pixels on the border of the image, g_P^f is specified by the single value $f(P)$. For the pixels having value 0.2, g_P^f is defined by the pair of values (0.2, 0.1), except for the lower left-hand 0.2, where g_P^f is defined by (0.2, 0). Finally, for the center pixel, g_P^f is defined by the triple of values (0.3, 0.2, 0). This implies that, as shown in Figure 2b, all the pixels having value 0.2, except for the lower left-hand one, belong to the fuzzy medial axis (since the center pixel's g_P^f is not $\geqslant$ their g_P^f's); thus 8 of the 25 pixels belong to the fuzzy medial axis. Thus specifying the FMAT requires 17 membership values (one disk requires three values, and seven require two each). This result remains true if we define the FMAT using only convex fuzzy disks, since the disks in this example are all convex. Note that specifying the image itself requires only 25 values.

For real images, the situation can be even worse. Figures 3a and 4a show, respectively, a 16×41 chromosome image and a 36×60 image of an 'S'; each of them has 32 possible membership values. In the first case all but 189 of the 656 pixels belong to the fuzzy medial axis (Figure 3b), and in the second case all but 411 of the 2160 pixels belong to it (Figure 4b). The results are similar if we allow only convex fuzzy disks in the FMAT; in the first case we still need all but 211 of the pixels (Figure 3c), and in the second case all but 730 (Figure 4c). [The fact that we need fewer maximal convex fuzzy disks, even though their values are smaller than those of the maximal fuzzy disks (the g_P^f's), is apparently because the convex fuzzy disks have fewer nonstrict local maxima.]

.1	.1	.1	.1	.1
.1	.2	.2	.2	.1
.1	.2	.3	.2	.1
.1	.2	.2	.2	.1
0	.1	.1	.1	.1

(a)

•	•	•	•	•
•	X	X	X	•
•	X	X	X	•
•	•	X	X	•
•	•	•	•	•

(b)

Figure 2. (a) 5×5 digital image. (b) X's denote pixels belonging to the fuzzy medial axis of (a).

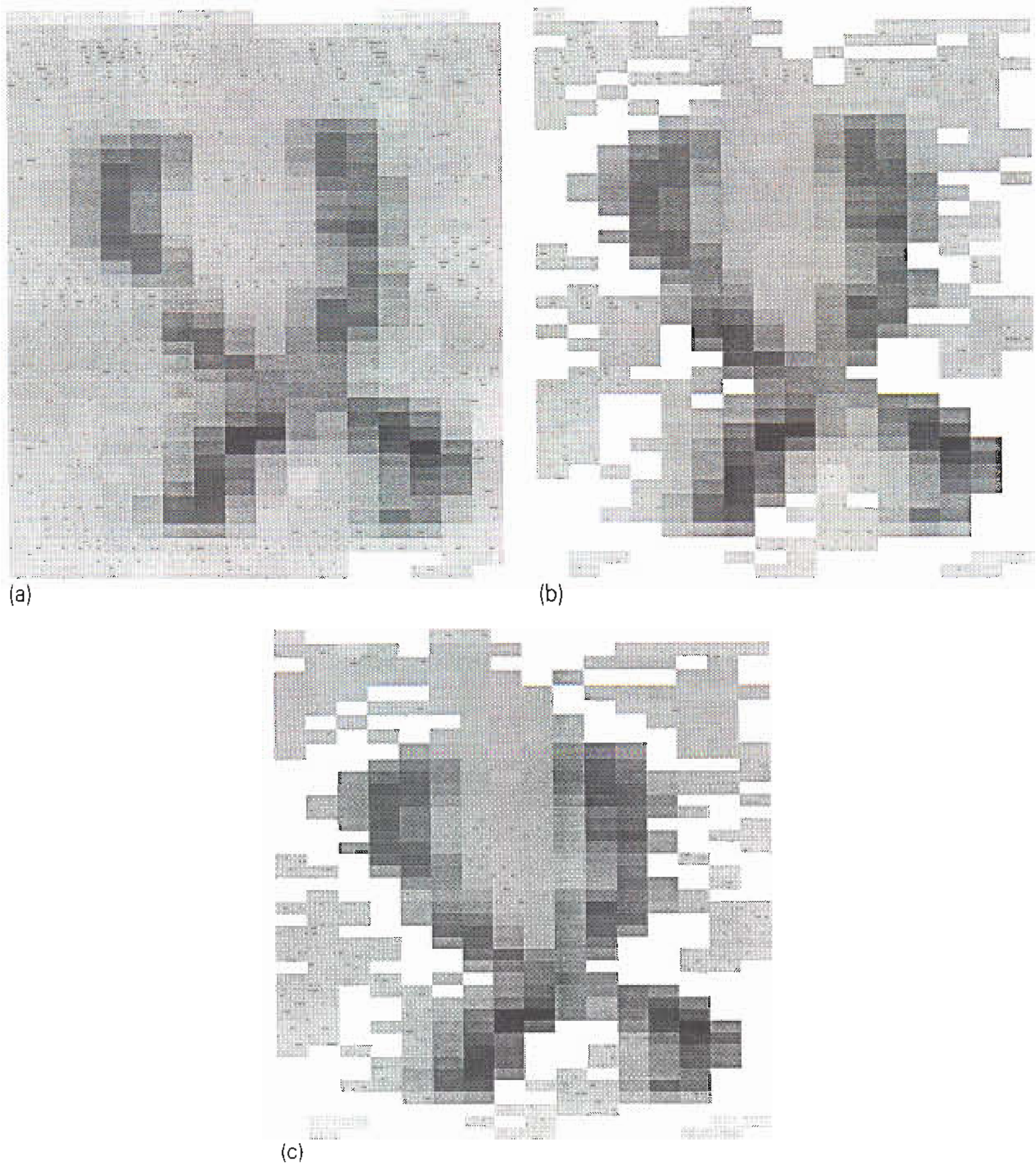

Figure 3. (a) 41 × 16 chromosome input. (b) Complete FMAT output. (c) FMAT output assuming convex disks.

Somewhat better results can be obtained by defining the FMAT using fuzzy disks of bounded radius (i.e., disks whose memberships are 0 beyond a given radius r); evidently, for any $r \geqslant 0$ the image is still the sup of these disks. (Of course, for $r = 0$ the FMAT is just the entire set of pixels in the image.) As we decrease r, the number of disks needed will increase, but since small disks are specified by fewer values, the total number of values needed may decrease. Unfortunately, for the images in Figures 3 and 4 it turns out that for every value of r, the number of values needed to specify the FMAT is at least as great as the number of pixels in the images, as shown in Tables 1 and 2.

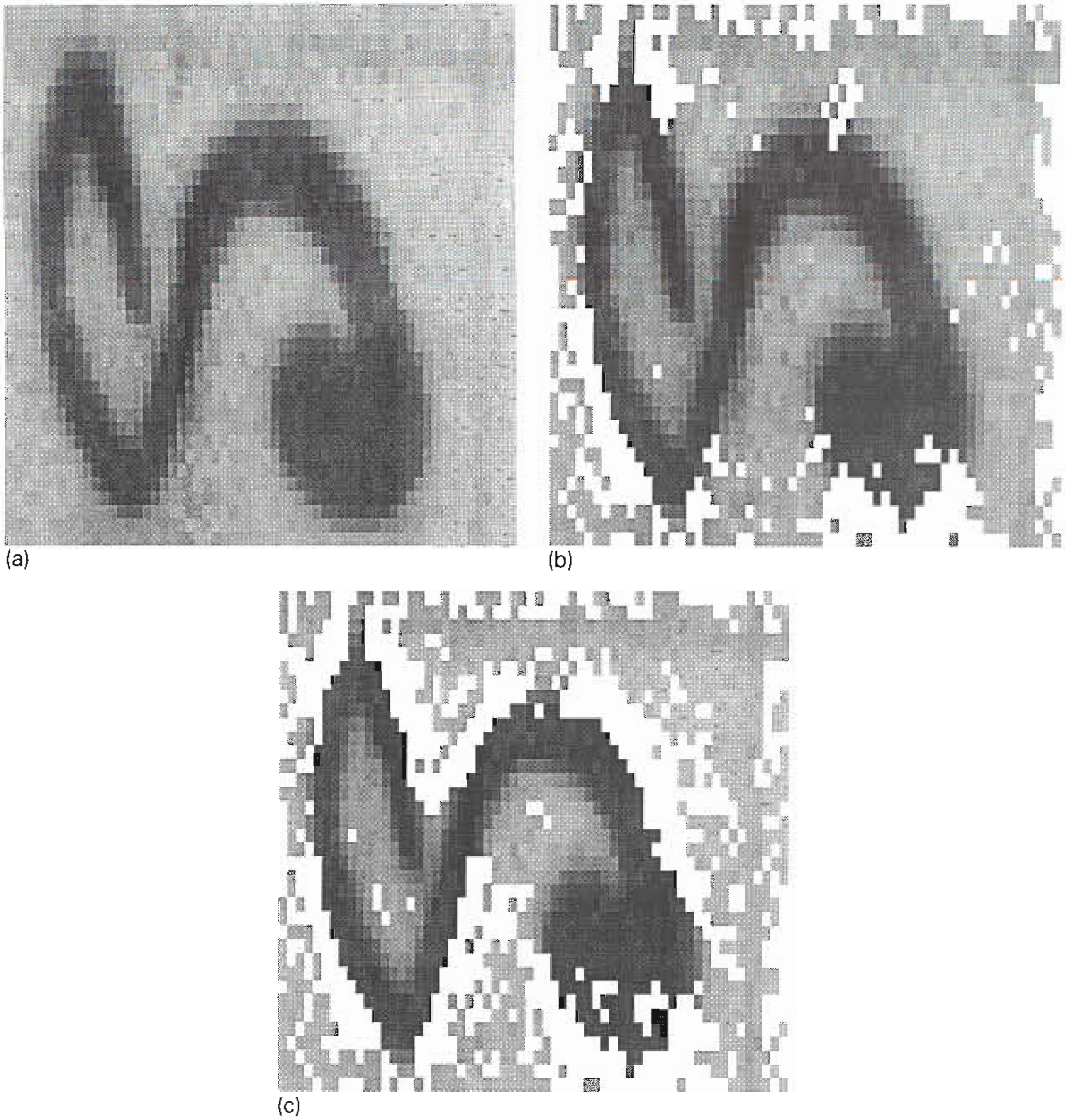

Figure 4. (a) 36×60 S input. (b) Complete FMAT output. (c) FMAT output assuming convex disks.

4. Concluding remarks

The FMAT, defined using either fuzzy disks or convex fuzzy disks, is a natural generalization of the MAT. Unfortunately, since O(n) membership values are required to specify a fuzzy disk in an $n \times n$ digital image, the FMAT is a compact representation of the image only if it involves a relatively small number of fuzzy disks.

The compactness of the representation can be improved in two ways:

(a) The FMAT, like the MAT, is redundant; g_P^f is not used if there exists a Q such that $g_P^f \leqslant g_Q^f$, but we could also eliminate many other g_P^f's for

Table 1
Number of disks and number of values needed for the chromosome image (Figure 3: 656 pixels) when we use disks of radii $\leqslant 7$, 5, 3, 2, 1, or 0

Radius	Disks	Values
7	304	1237
5	310	1193
3	311	980
2	319	828
1	345	656
0	656	656

Table 2
Number of disks and number of values needed for the *S* image (Figure 4: 2160 pixels) when we use disks of radii $\leqslant 17$, 13, 10, 7, 5, 3, 1, or 0

Radius	Disks	Values
17	1744	15,278
13	1749	14,678
10	1752	13,411
7	1764	11,251
5	1817	9,440
3	1903	7,074
1	2059	4,031
0	2160	2,160

which there exist *sets* $\mathcal{Q}$ of Q's such that $g_P^f \leqslant \sup_{\mathcal{Q}} g_Q^f$. On this approach to reducing the redundancy of the MAT see [9].

(b) The MAT and FMAT completely determine the original image. For many purposes it would suffice to use a representation from which an approximation to the image could be constructed. A MAT-based approach of this type is described in [10].

It would be of interest to generalize both of these approaches to the FMAT.

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