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Automatic grey level thresholding through index of fuzziness and entropy

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Abstract: Algorithms for automatic thresholding of grey levels (without reference to histogram) are described using the terms 'index of fuzziness' and 'entropy' of a fuzzy set. Their values are seen to be minimum when the crossover point of an S-function corresponds to boundary levels among different regions in image space. The effectiveness of the algorithms is demonstrated for images having both bimodal and multimodal grey level distributions.

Key words: Pattern recognition, image processing, fuzzy sets.

1. Introduction

The problem of grey level thresholding plays an important role in image processing. For example, in enhancing contrast in an image we need to select proper threshold levels from its histogram so that some suitable non-linear transformation can highlight a desirable set of pixel intensities compared to others. Similarly, in image segmentation one needs proper histogram thresholding whose objective is to establish boundaries in order to partition the image space into meaningful regions.

The present work illustrates an application of theory of fuzzy sets to make this task automatic so that an optimum threshold (or set of thresholds) may be estimated without the need to refer directly to the histogram. These are explained by the terms 'index of fuzziness' (Kaufmann (1975)) and 'entropy' (De Luca and Termini (1972)) of a fuzzy set. Since these terms reflect the measures of closeness of a grey tone image to its two-tone version, they provide a quantitative measure (Pal (1982)) of image ambiguity when the cross-over point is set to a predetermined value. Modification of cross-over point will result in variation of these parameters and so a set of minima is obtained corresponding to the optimum threshold levels.

2. Definitions

Let $X = \{\mu_X(x_{mn}) = \mu_{mn}/x_{mn},\quad m = 1, 2, \ldots, M;\ n = 1, 2, \ldots, N\}$ be the fuzzy set representation of the pattern corresponding to an $M \times N$, L-level image array, where $\mu_X(x_{mn})$ or μ_{mn}/x_{mn} $(0 \le \mu_{mn} \le 1)$ denotes the grade of possessing some property μ_{mn} (as defined in the next section) by the (m, n)th pixel intensity x_{mn}. Let $\boldsymbol{X} = \{\mu_{\boldsymbol{X}}(x_{mn})\}$ be similarly defined as the nearest ordinary plane to X, such that $\mu_{\boldsymbol{X}}(x_{mn}) = 0$ if $\mu_X(x_{mn}) \le 0.5$ and is equal to 1 for $\mu_X(x_{mn}) > 0.5$.

The linear index of fuzziness $\nu_\ell(X)$, quadratic

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index of fuzziness $v_q(X)$ and entropy $H(X)$ of the image X are defined as (Pal (1982))

$$v_\ell(X) = \frac{2}{MN} \sum_m \sum_n |\mu_X(x_{mn}) - \mu_{\mathbf{X}}(x_{mn})| \quad (1a)$$

$$= \frac{2}{MN} \sum_m \sum_n \mu_{X \cap \bar{X}}(x_{mn}); \quad (1b)$$

$$v_q(X) = \frac{2}{\sqrt{MN}} \left[\sum_m \sum_n (\mu_X(x_{mn}) - \mu_{\mathbf{X}}(x_{mn}))^2 \right]^{1/2} \quad (2)$$

and

$$H(X) = \frac{1}{MN \ln 2} \sum_m \sum_n \mathrm{Sn}(\mu_X(x_{mn})) \quad (3a)$$

with the Shannon's function

$$\mathrm{Sn}(\mu_X(x_{mn})) = -\mu_X(x_{mn}) \ln \mu_X(x_{mn}) - (1-\mu_X(x_{mn})) \ln (1-\mu_X(x_{mn}))$$

$$m = 1, 2, \ldots, M; \qquad n = 1, 2, \ldots, N. \quad (3b)$$

$v(X)$ measures the distance (linear for $v_\ell(X)$ and quadratic for $v_q(X)$) between the fuzzy property plane X and its nearest ordinary plane $\mathbf{X}$. $X \cap \bar{X}$ is the intersection between fuzzy image planes $X = \{p_{mn}/x_{mn}\}$ and $\bar{X} = \{(1-p_{mn})/x_{mn}\}$, the complement of X. $\mu_{X \cap \bar{X}}(x_{mn})$ denotes the degree of membership of x_{mn} to such a property plane $X \cap \bar{X}$ so that

$$\mu_{X \cap \bar{X}}(x_{mn}) = \mu_{mn} \cap \bar{\mu}_{mn} = \min\{\mu_{mn}, (1-\mu_{mn})\}$$

for all (m, n).

The term 'entropy' on the other hand, uses the Shannon's function in the property plane but its meaning is quite different from that of the classical entropy because no probabilistic concept is needed to define it.

3. Property plane and selection of threshold

To obtain the μ_{mn} plane from the spatial x_{mn} plane of the image X, let us consider the standard S-function which has the form (Zadeh (1975))

$$\mu_X(x_{mn}) = S(x_{mn}; a, b, c)$$

$$= 0; \quad x_{mn} \le a, \quad (4a)$$

$$= 2[(x_{mn}-a)/(c-a)]^2, \quad a \le x_{mn} \le b, \quad (4b)$$

$$= 1 - 2[(x_{mn}-c)/(c-a)]^2, \quad b \le x_{mn} \le c, \quad (4c)$$

$$= 1; \quad x_{mn} \ge c \quad (4d)$$

with

$$b = \tfrac{1}{2}(a+c) \quad \text{and} \quad b - a = c - b = \triangle b.$$

The parameter b is the cross-over point, i.e., $S(b; a, b, c) = 0.5$. $\triangle b$ is the bandwidth. This is explained in Fig. 1.

For a particular cross-over point, say, $b = l_c$ we have $\mu_X(l_c) = 0.5$ and the μ_{mn} plane would contain values >0.5 or <0.5 corresponding to $x_{mn} > l_c$ or $< l_c$. The above three terms (eqs. (1) to (3)) then measure the average ambiguity in X by computing $\mu_{X \cap \bar{X}}(x_{mn})$ or $\mathrm{Sn}(\mu_X(x_{mn}))$ which is 0 if $\mu_X(x_{mn}) = 0$ or 1 and is maximal for $\mu_X(x_{mn}) = 0.5$. The v or H value would therefore increase monotonically in the interval [0, 0.5] and decrease monotonically in [0.5, 1] with a maximum at $\mu = 0.5$ in the μ_{mn} plane of X.

The selection of a cross-over point at $b = l_c$ implies the allocation of the grey levels $< l_c$ and $> l_c$ within the two clusters namely, background and object of a bimodal image. The contribution of the levels towards $v(X)$ and $H(X)$ is mostly from those around l_c and would decrease as we move away from l_c. Again, since the nearest ordinary plane $\mathbf{X}$ (which gives the two-tone version of X) is depen-

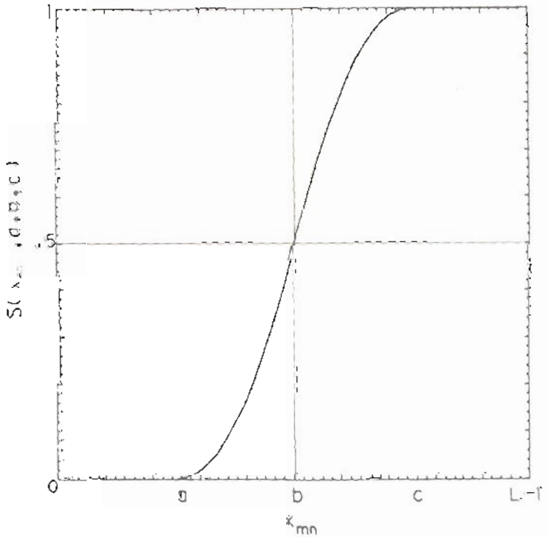

Fig. 1. Standard S function.

dent on the position of cross-over point, a proper selection of b may therefore be obtained which will result in appropriate segmentation of object and background. In other words, if the grey level of image X has bimodal distribution, then the above criteria for different values of b would result in a minimum v or H value only when b corresponds to the appropriate boundary between the two clusters.

For such a position of the threshold (cross-over point), there will be a minimum number of pixel intensities in X having $\mu_{mn} \simeq 0.5$ (resulting in v or $H \simeq 1$) and maximum number of pixel intensities having $\mu_{mn} \simeq 0$ or 1 (resulting in v or $H \simeq 0$) thus contributing least towards $v(X)$ or $H(X)$. This optimum (minimum) value would be greater for any other selection of the cross-over point.

This suggests that modification of the cross-over point will result in variation of the parameters $v(X)$ and $H(X)$ and so an optimum threshold may be estimated for automatic histogram thresholding problems without the need to refer directly to the histogram of X. The above concept can also be extended to an image having multimodal distribution in grey levels in which one would have several minima in v and H values corresponding to different threshold points in the histogram.

4. Implementation

Figs. 2 and 3 show images of 96×99, 32-level handwritten script 'shu' and an 128×145, 256-level radiograph of a part of wrist respectively together with their histograms. Table 1 illustrates the values of v and H corresponding to Fig. 2 (which has a bimodal histogram) for different values of the cross-over point when Δb ($= b - a = c - A$) of the S-function is considered to be 4, 6, 8 and 10. The purpose of this experiment is to demonstrate the effect of the parameters a and c (of S-function in Fig. 1) in selecting thresholds. The parameters v and H are seen to possess minimum around 16.5 showing the threshold level between background and object.

For Fig. 3 (having a multimodal histogram) we have considered only the parameter 'linear index of fuzziness' $v_\ell(X)$ and its variation with b for dif-

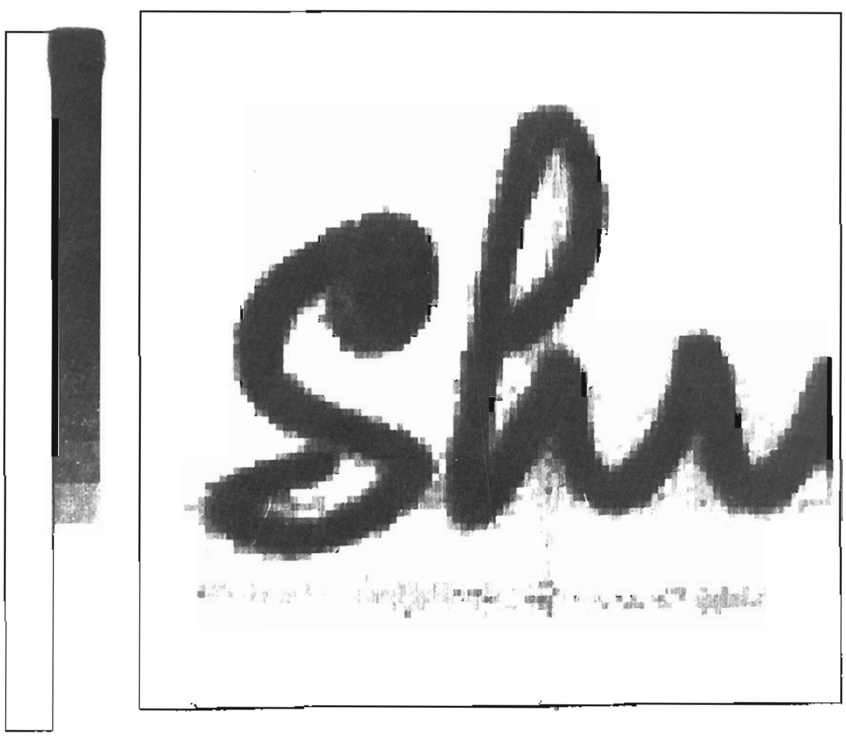

Fig. 2a. Bimodal image.

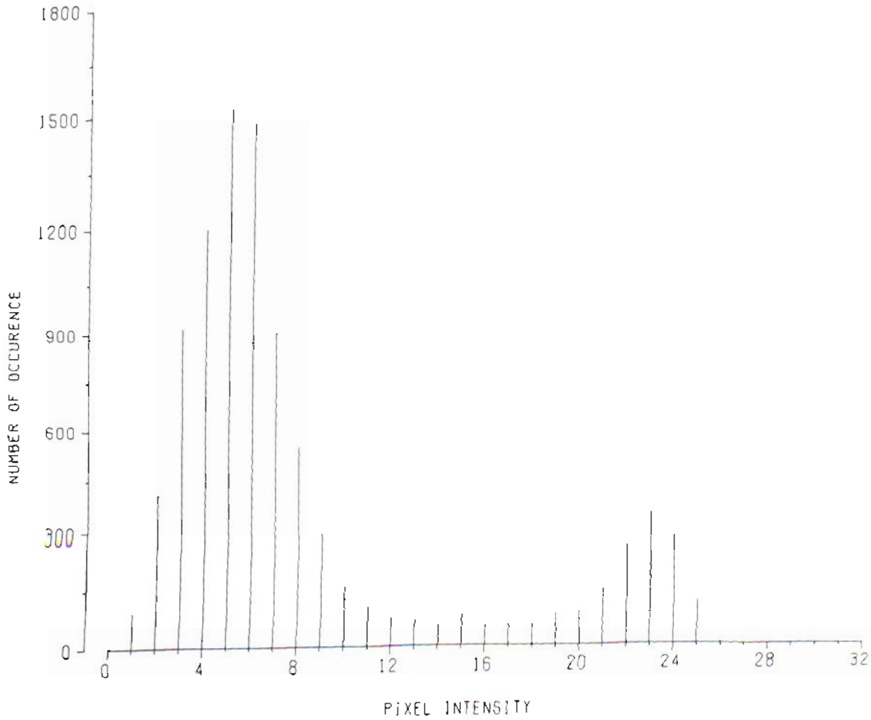

Fig. 2b. Histogram of Fig. 2a.

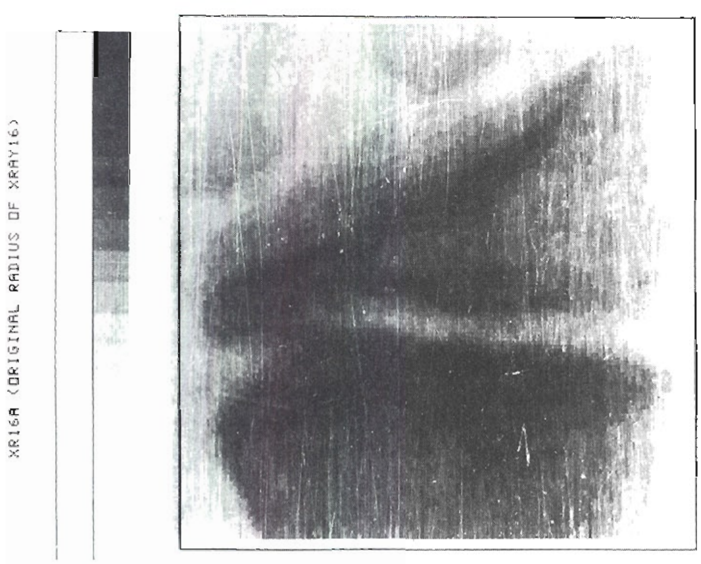

Fig. 3a. Multimodal image.

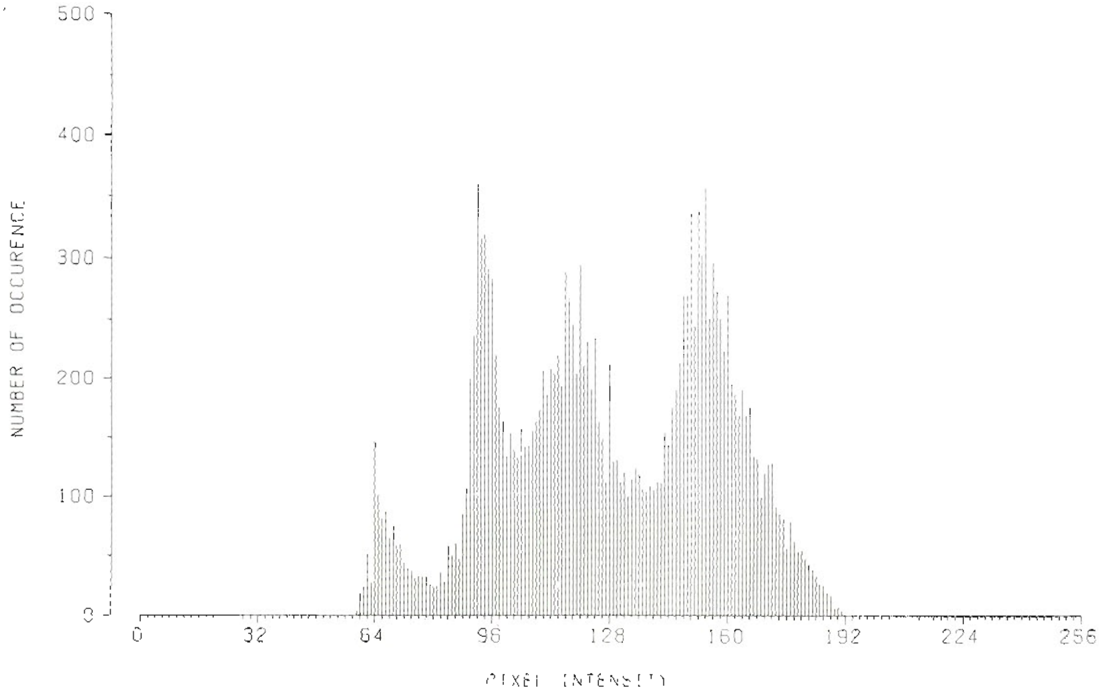

Fig. 3b. Histogram of Fig. 3a.

Table 1
$v_f(X)$, $v_q(X)$ and $H(X)$ for different values of b

$\triangle b$	Cross-over point (b)							
	7.5	10.5	13.5	15.5	16.5	17.5	20.5	22.5
	0.2186	0.0479	0.0193	0.0179	0.0174	0.0187	0.0405	0.0722
4.0	0.3503	0.1548	0.1029	0.1026	0.0973	0.1015	0.1450	0.2158
	0.3560	0.0861	0.0321	0.0287	0.0291	0.0313	0.0695	0.1054
	0.3264	0.0974	0.0330	0.0281	0.0284	0.0318	0.0649	0.0915
6.0	0.4399	0.2093	0.1316	0.1272	0.1246	0.1298	0.1889	0.2539
	0.5102	0.1867	0.0580	0.0463	0.0481	0.0553	0.1059	0.1249
	0.4124	0.1669	0.0561	0.0423	0.0436	0.0496	0.0842	0.1045
8.0	0.5087	0.2678	0.1601	0.1495	0.1490	0.1568	0.2236	0.2788
	0.6150	0.3207	0.1096	0.0753	0.0784	0.0894	0.1295	0.1372
	0.4773	0.2414	0.0967	0.0663	0.0650	0.0697	0.0992	0.1145
10.0	0.5624	0.3269	0.1936	0.1734	0.1742	0.1841	0.2505	0.2966
	0.6804	0.4466	0.2045	0.1314	0.1233	0.1255	0.1459	0.1470

Upper score: $v_f(X)$. Middle score: $v_q(X)$. Lower score $H(X)$.

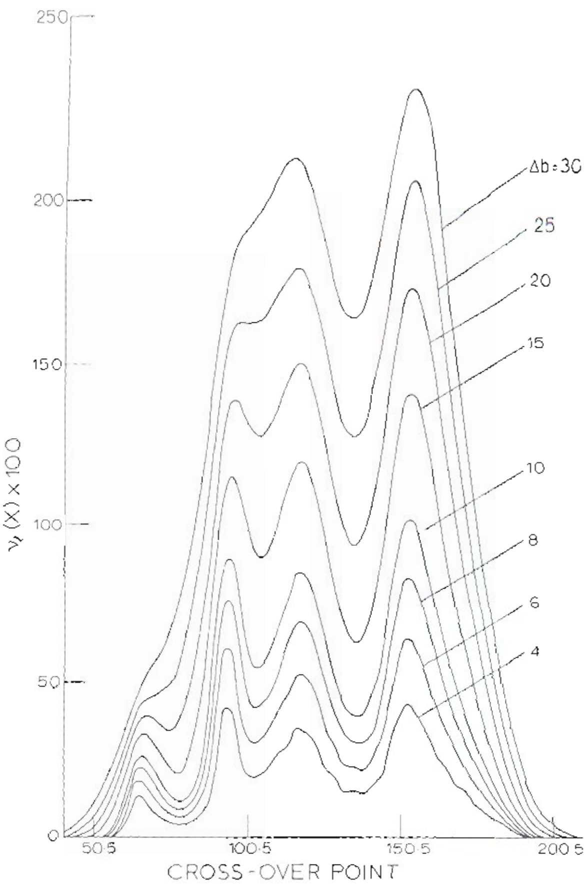

Fig. 4. Variation of $\nu_\ell(X)$ with b.

ferent values of $\triangle b$. This is graphically shown in Fig. 4 in which a set of minima around 78.5, 102.5 and 136.5 are observed corresponding to boundary levels between flesh and different regions of bones. At low values of $\triangle b$ (~4) the curve shows slight irregularity giving some local minima. Higher values of $\triangle b$ (~8 to 20) give 3 clearly defined minima. At still higher values, where $\triangle b$ exceeds half the interval between histogram peaks (~25, 30) some of the minima are lost. It appears from Table 1 and Fig. 4 that the value of $\triangle b$ is not critical.

5. Conclusions

A method has been outlined using the concept of fuzzy sets whereby a satisfactory choice of threshold level on image segmentation may be determined without the need for histogram determination.

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