# WELFARE ANALYSIS OF PRIVATIZATION IN A BILATERAL MONOPOLY

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**Abstract**

The paper analyzes the welfare implications of privatization in a bilateral monopoly. We characterize and compare the behavior of optimum prices, profit and welfare with privatization in the upstream/input sector and downstream/output sector. A comparison across the two cases shows that even if the marginal cost of production in the vertical stream is the same, welfare is not necessarily identical. Further, welfare with privatization upstream is less than welfare with privatization downstream if the weight of consumer surplus in the welfare is not very large. If the marginal cost of production is not identical for the two cases, the welfare results from privatization upstream or downstream may go either way. Hence we conclude that depending on the environment, an argument can be made for public policy that is directed at privatizing either the upstream or the downstream sector.

## 1 Introduction

There are markets that were traditionally dominated by firms with monopoly power in input and output sectors. In this framework it is well known that, depending on the environment, vertical restraints can either increase or decrease welfare. So for instance a vertically integrated bilateral monopoly is more profitable than two separate firms and consumers are better off because they face a lower price. Thus welfare is unambiguously increased from the elimination of double marginalization. On the other hand, in a vertical framework with downstream retail competition, an upstream monopoly manufacturer can increase profit by using vertical restraints such as exclusive territories or franchise fees and the welfare analysis can be shown to be ambiguous. See for instance, Tirole (1988) for discussion. This strand of research almost uniformly assumes that firms in both sectors, either manufacturers of an intermediate and final good or in a manufacturer/retailer relationship, are private firms.

However, there are examples of countries where public companies operate in a bilateral monopoly framework. In some markets, core infrastructure sectors such as electric power generation or steel production were typically public firms as also were key downstream sectors such as rail and air transportation that employ the upstream input.

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In a one sector model, Matsumura (1998) studies a quantity setting mixed oligopoly, where the private firm maximizes profit and the public firm considers both profit and social welfare. He finds that neither full privatization nor full nationalization is optimal under moderate conditions. In a three sector framework with information uncertainty, Scheinkman and Glaeser (1996) use a model which has an upstream firm, a downstream firm and a retailer and they assume that privatization creates information benefits. Interestingly, they show that if only one sector can be privatized and demand uncertainty is high, it is possible that the optimal policy involves privatizing upstream. In general, they conclude that across industries, those with large amounts of cost and/or demand uncertainties should be privatized first. It is frequently difficult to privatize both sectors, usually for political reasons. In view of this, if only one sector can be privatized, which should it be? We focus on the welfare implications of privatization in a bilateral monopoly and are interested in whether public policy should adopt a hands off approach to privatization in this framework or whether an argument exists for policy directed at privatizing specifically the upstream or downstream sector.

It is accepted that a key benefit from privatization is an improvement in productivity and we retain this assumption in our analysis.[1] We assume welfare is a weighted average of consumer surplus and profit of the private and public firms. We characterize and compare the behavior of optimum prices, profit and welfare with privatization in the upstream/input sector and also privatization in the downstream/output sector. In studying and comparing welfare across the two cases, we find that even if the marginal cost of production in the vertical stream is the same, welfare is not necessarily identical. Further, welfare with privatization upstream is less than welfare with privatization downstream if the weight of consumer surplus in the welfare is not very large. If the marginal cost of production is not identical for the two cases, the welfare results from privatization upstream or downstream may go either way. Hence we conclude that depending on the environment, an argument can be made for public policy that is directed at privatizing either the upstream or the downstream sector.

The paper is organized as follows: Section 2 presents the Model. Sections 3.1 and 3.2 provide the solutions while Section 3.3 presents welfare comparisons. Section 4 concludes the paper.

## 2 Model

The market consists of a bilateral monopoly where an upstream firm $(u)$ produces the input used by a downstream firm $(d)$ to produce the final output. The bilateral monopoly consists of a public firm $G$ and a private firm $P$. The public firm $G$ maximizes welfare and the private firm $P$ maximizes profit. We assume welfare is a weighted average of the sum of consumer surplus and profits of both firms and is given by:

$$W = \alpha_p \pi_p + \alpha_g \pi_g + \alpha_c CS.$$

where

$$\pi_p = \text{profit of private firm} \quad (1)$$

$$\pi_g = \text{profit of public firm} \quad (2)$$

$$CS = \text{consumer surplus} \quad (3)$$

and more explicit description of $\pi_p$, $\pi_g$ and $CS$ are given below.

The sum of the weights of profits and consumer surplus must equal 1. This is a normalizing restriction to compare across alternative scenarios. The additional assumptions on the model in the two cases will be

[1]In an oligopoly model, De Fraja and Delbono (1989) show that welfare might actually be higher if the public firm maximizes profit but they do not consider the effect of higher productivity with privatization. They argue that privatization is better even if it does not increase productivity.

mentioned later.

**Assumption 1.** $\alpha_p + \alpha_g + \alpha_c = 1$.

We assume fixed proportions production with one unit of input used to produce one unit of output. The market demand downstream is linear and of the form: $q_d = 1 - p_d$.[2] The consumer surplus $CS$ is then given by

$$CS = \int_{p_d}^{1} (1-t)dt = \frac{(1-p_d)^2}{2}$$

where $p_d$ is the price charged by the downstream firm for the final output.

Let $c_i$ be the marginal cost of production for public firm $i, i = u, d$. The marginal cost of production for a private firm is $\theta_i c_i$, where $0 < \theta_i < 1$, $i = u, d$ represents the productivity efficiency parameter from privatization.

The model is a two-stage game. In stage one, the upstream firm announces an input price, $p_u$. In the second stage, the downstream firm learns the input price and determines the downstream price, $p_d$. The market clears at the end of stage two.

Case 1 analyzes the bilateral monopoly with an upstream private firm and a downstream public firm. Case 2 is the reverse, with privatization downstream, and a public firm in the upstream input market. We now describe the model for the two cases.

**Case 1**. In this case, the upstream firm is private and the downstream firm is public. Then,

$$\pi_g = (p_d - p_u - c_d)(1 - p_d) \text{ and } \pi_p = (p_u - \theta_u c_u)(1 - p_d) \tag{4}$$

where

$$0 < p_u - \theta_u c_u < p_u < 1 \text{ and } 0 < p_d - p_u - c_d < p_d < 1. \tag{5}$$

The downstream public firm will maximize welfare:

$$\max_{p_d} W(p_u, p_d) = \max \alpha_p \pi_p + \alpha_g \pi_g + \alpha_c \int_{p_d}^{1} (1-t)dt. \tag{6}$$

The upstream private firm will maximize profit:

$$\max_{p_u} \pi_p. \tag{7}$$

**Case 2**. In this case, the upstream firm is public and the downstream firm is private. Then

$$\pi_g = (p_u - c_u)(1 - p_d) \text{ and } \pi_p = (p_d - p_u - \theta_d c_d)(1 - p_d). \tag{8}$$

where

$$0 < p_u - c_u < p_u < 1 \text{ and } 0 < p_d - p_u - \theta_d c_d < p_d < 1. \tag{9}$$

The downstream firm will maximize

$$\max_{p_d} \pi_p. \tag{10}$$

The upstream public firm will maximize welfare:

$$\max_{p_u} W(p_u, p_d) = \alpha_p \pi_p + \alpha_g \pi_g + \alpha_c \int_{p_d}^{1} (1-t)dt. \tag{11}$$

[2]The linear demand formulation keeps the analysis tractable and yet yields a rich set of results.

# 3 Results

## 3.1 Case 1: Privatization upstream

With the private monopoly upstream and the public monopoly downstream, Proposition 1 summarizes the solutions to the maximization problem in Case 1. Assumptions 1 and 2 below state conditions for nonnegative outcomes (specified in (5)) and for second order conditions in Proposition 1 to be satisfied. The proof will clarify the conditions as needed.

Assumption 2 (i), (ii), (iii) and (iv) ensure respectively that the optimum $(p_u,\ p_d)$ satisfies (i) $p_u - \theta_u c_u > 0$, (ii) $0 < p_d - p_u - c_d$ (iii) $p_u < 1$ and (iv) $p_d < 1$.

**Assumption 2.**

(i) $\alpha_g > \alpha_p$.

(ii) $2\alpha_g^2 - 4\alpha_g\alpha_p - \alpha_g - 2\alpha_p^2 + 2\alpha_p > 0$.

(iii) $(\alpha_g - 2\alpha_p)(1 - \theta_u c_u) + \alpha_g c_d > 0$ and $3\alpha_g + \alpha_p > 1$.

(iv) $0 < \theta_u c_u + c_d < 1$.

*Remark.* It may be noted that since $\alpha_p > \alpha_p^2$,

$$2\alpha_g^2 - 4\alpha_g\alpha_p - \alpha_g - 2\alpha_p^2 + 2\alpha_p > \alpha_g(2\alpha_g - 4\alpha_p - 1).$$

Hence a *sufficient* condition for Assumption 2 (i)–(iii) to hold is

$$2\alpha_g > 4\alpha_p + 1. \tag{12}$$

In other words, Assumption 2 (i)–(iii) holds if $\alpha_g$ is large and $\alpha_p$ is small. Note that this condition implies, among others, that the public firm does not have a soft budget and is not subsidized even though it is maximizing welfare. We believe this is an important component of the analysis, especially for a welfare maximizing firm.

**Proposition 1.** *Consider a bilateral monopoly with a private firm upstream and a public firm downstream. The private firm maximizes profit and the public firm maximizes welfare. Suppose Assumptions 1 and 2 hold. Then for every fixed $p_u$, the maximum value of $W(p_u, p_d)$ is obtained at a unique $p_d = p_d(p_u)$. The maximum value of $W(p_u, p_d(p_u))$ is obtained at a unique $p_u = p_u^0$. Further,*

$$p_u^0 = \theta_u c_u + \frac{\alpha_g}{2(\alpha_g - \alpha_p)}(1 - \theta_u c_u - c_d) \tag{13}$$

$$p_d^0 \equiv p_d(p_u^0) = 1 - \frac{\alpha_g(1 - \theta_u c_u - c_d)}{2(3\alpha_g + \alpha_p - 1)} \tag{14}$$

$$W^{PU} \equiv W(p_u^0, p_d^0) = \frac{\alpha_g^2}{8(3\alpha_g + \alpha_p - 1)}(1 - \theta_u c_u - c_d)^2. \tag{15}$$

**Proof**.

$$\frac{\partial W}{\partial p_d} = \alpha_g(1 - p_d) - [\alpha_p(p_u - \theta_u c_u) + \alpha_g(p_d - p_u - c_d)] - \alpha_c(1 - p_d) \tag{16}$$

$$= (1 - p_d)(\alpha_g - \alpha_c) - [\alpha_p(p_u - \theta_u c_u) + \alpha_g(p_d - p_u - c_d)] \tag{17}$$

$$\frac{\partial W}{\partial p_d} = 0 \Rightarrow (1 - p_d)(\alpha_g - \alpha_c) = \alpha_p(p_u - \theta_u c_u) + \alpha_g(p_d - p_u - c_d) \tag{18}$$

$$\frac{\partial^2 W}{\partial p_d^2} = -(\alpha_g - \alpha_c) - \alpha_g = -(3\alpha_g + \alpha_p - 1) < 0, \text{ using Assumption 2 (iii).} \tag{19}$$

Suppose $p_d = p_d(p_u)$ is the solution of (18). Equation (18) implies

$$p_d[\alpha_g - \alpha_c + \alpha_g] = (\alpha_g - \alpha_c) + (\alpha_g - \alpha_p)p_u + [\alpha_p\theta_u c_u + \alpha_g c_d].$$

Hence

$$\frac{\partial p_d}{\partial p_u} = \frac{\alpha_g - \alpha_p}{2\alpha_g - \alpha_c} > 0 \text{ using Assumptions 2 (i) and (iii).}$$

Now consider $\pi_p$ where $p_d = p_d(p_u)$ is the solution of (18). Using (8),

$$\begin{aligned} \frac{\partial \pi_p}{\partial p_u} &= 1 - p_d + (p_u - \theta_u c_u)(-\frac{\partial p_d}{\partial p_u}) && (20)\\ &= (1 - p_d) - (p_u - \theta_u c_u)\frac{\alpha_g - \alpha_p}{2\alpha_g - \alpha_c} && (21)\end{aligned}$$

$$\frac{\partial^2 \pi_p}{\partial p_u^2} = -\frac{\partial p_d}{\partial p_u} - \frac{\alpha_g - \alpha_p}{2\alpha_g - \alpha_c} < 0 = -2\frac{\partial p_d}{\partial p_u} < 0. \tag{22}$$

Hence the maximum value of $\pi^u$ is obtained by solving $\frac{\partial \pi^u}{\partial p_u} = 0$, which implies, by (20),

$$1 - p_d = (p_u - \theta_u c_u)\frac{\alpha_g - \alpha_p}{2\alpha_g - \alpha_c}. \tag{23}$$

Now using (18) and putting everything back into the $W$, we get

$$\begin{aligned} W &= (1 - p_d)[\alpha_p(p_u - \theta_u c_u) + \alpha_g(p_d - p_u - c_d)] + \alpha_c\int_{p_d}^1 (1 - t)dt && (24)\\ &= (1 - p_d)[(1 - p_d)(\alpha_g - \alpha_c)] + \frac{\alpha_c}{2}(1 - p_d)^2 && (25)\\ &= \frac{2\alpha_g - \alpha_c}{2}(1 - p_d)^2. && (26)\end{aligned}$$

We now use equations (18) and (23) to obtain

$$\begin{aligned} p_d &= 1 - \frac{(p_u - \theta_u c_u)(\alpha_g - \alpha_p)}{2\alpha_g - \alpha_c} \text{ and} && (27)\\ p_d &= \frac{\alpha_g - \alpha_c + p_u(\alpha_g - \alpha_p) + (\alpha_p\theta_u c_u + \alpha_g c_d)}{2\alpha_g - \alpha_c}. && (28)\end{aligned}$$

The above implies, upon simplification,

$$p_u - \theta_u c_u = \frac{\alpha_g}{2(\alpha_g - \alpha_p)}(1 - \theta_u c_u - c_d) \tag{29}$$

and hence using (23), (26) and (29),

$$\begin{aligned} W^{PU} &= \frac{2\alpha_g - \alpha_c}{2}(p_u - \theta_u c_u)^2\frac{(\alpha_g - \alpha_p)^2}{(2\alpha_g - \alpha_c)^2} && (30)\\ &= \frac{2\alpha_g - \alpha_c}{2}\left[\frac{\alpha_g(1 - \theta_u c_u - c_d)}{2(\alpha_g - \alpha_p)}\right]^2\frac{(\alpha_g - \alpha_p)^2}{(2\alpha_g - \alpha_c)^2} && (31)\\ &= \frac{\alpha_g^2}{8(2\alpha_g - \alpha_c)}(1 - \theta_u c_u - c_d)^2 && (32)\\ &= \frac{\alpha_g^2}{8(3\alpha_g + \alpha_p - 1)}(1 - \theta_u c_u - c_d)^2. && (33)\end{aligned}$$

From (29), it is clear that $p_u - \theta_u c_u > 0$ if $\alpha_g > \alpha_p$ and $1 - \theta_u c_u - c_d > 0$.

Moreover,

$$
\begin{aligned}
p_u < 1 \quad &\Leftrightarrow \quad \theta_u c_u + \frac{\alpha_g}{2(\alpha_g - \alpha_p)}(1 - \theta_u c_u - c_d) < 1 && (34)\\
&\Leftrightarrow \quad 2(\alpha_g - \alpha_p)\theta_u c_u + \alpha_g(1 - \theta_u c_u - c_d) < 2(\alpha_g - \alpha_p) && (35)\\
&\Leftrightarrow \quad (\alpha_g - 2\alpha_p)(1 - \theta_u c_u) + \alpha_g c_d > 0. && (36)
\end{aligned}
$$

Further, using (23), and (29),

$$
\begin{aligned}
1 - p_d &= \frac{\alpha_g - \alpha_p}{2\alpha_g - \alpha_c}(p_u - \theta_u c_u) && (37)\\
&= \frac{(\alpha_g - \alpha_p)}{2\alpha_g - \alpha_c}\frac{\alpha_g}{2(\alpha_g - \alpha_p)}(1 - \theta_u c_u - c_d) && (38)\\
&= \frac{\alpha_g}{2(3\alpha_g + \alpha_p - 1)}(1 - \theta_u c_u - c_d). && (39)
\end{aligned}
$$

Hence

$$
\begin{aligned}
p_d - p_u - c_d \quad &> \quad 0 && (40)\\
&\Leftrightarrow \quad 1 - \frac{\alpha_g(1 - \theta_u c_u - c_d)}{2(3\alpha_g + \alpha_p - 1)} - \left[\theta_u c_u + \frac{\alpha_g}{2(\alpha_g - \alpha_p)}(1 - \theta_u c_u - c_d)\right] > c_d && (41)\\
&\Leftrightarrow \quad 1 - \theta_u c_u - c_d > \frac{\alpha_g(1 - \theta_u c_u - c_d)}{2}\left[\frac{1}{3\alpha_g + \alpha_p - 1)} + \frac{1}{\alpha_g - \alpha_p}\right] && (42)\\
&\Leftrightarrow \quad \frac{\alpha_g(4\alpha_g - 1)}{2(3\alpha_g + \alpha_p - 1)(\alpha_g - \alpha_p)} < 1 && (43)\\
&\Leftrightarrow \quad 2\alpha_g^2 - 4\alpha_g\alpha_p - \alpha_g - 2\alpha_p^2 + 2\alpha_p > 0. && (44)
\end{aligned}
$$

This proves the Proposition completely. ▲

**Behavior of the optimum prices.** From equation (13), it is clear that $\frac{\partial p_u^0}{\partial \alpha_p} > 0$. Further,

$$
\frac{\partial \log p_u^0}{\partial \alpha_g} = \frac{1}{\alpha_g} - \frac{1}{\alpha_g - \alpha_p} < 0.
$$

Using (14), it is also obvious that $\frac{\partial p_d^0}{\partial \alpha_p} > 0$.

Further,

$$
\frac{\partial \log p_d^0}{\partial \alpha_g} = -[\frac{1}{\alpha_g} - \frac{3}{3\alpha_g + \alpha_p - 1}] = \frac{1 - \alpha_p}{\alpha_g(3\alpha_g + \alpha_p - 1)} > 0.
$$

With the private firm upstream and the public firm downstream, input price is a decreasing function of $\alpha_g$ and an increasing function of $\alpha_p$, while the downstream price is an increasing function of both $\alpha_g$ and $\alpha_p$. In other words, when the weight of private profit in welfare is marginally higher, the upstream (private) firm increases its price. The downstream (public) firm thus has to pay higher input prices, and also does not have the option of a negative profit. Hence it follows by increasing its price. Conversely, when the weight of public profit marginally increases, the upstream (private) price is lower in anticipation of a higher downstream (public) price . These results are summarized in Finding 1.

**Finding 1.** Under Assumptions 1 and 2,

(i) $\dfrac{\partial p_u^0}{\partial \alpha_g} < 0$ and $\dfrac{\partial p_u^0}{\partial \alpha_p} > 0.$

(ii) $\dfrac{\partial p_d^0}{\partial \alpha_g} > 0$ and $\dfrac{\partial p_d^0}{\partial \alpha_p} > 0.$

**Behavior of optimum profit of the private firm upstream.** Going back to equation (24), and using the values of $(p_u^0 - \theta_u c_u)$ and $1 - p_d^0$, the optimized profit $\pi_p$ of the private firm is given by

$$\pi_p \equiv \pi_p(p_u^0, p_d^0) = (1 - p_d^0)(p_u^0 - \theta_u c_u) = \frac{\alpha_g^2}{2(3\alpha_g + \alpha_p - 1)2(\alpha_g - \alpha_p)}(1 - \theta_u c_u - c_d)^2.$$

It is clear that

$$\frac{\partial \log \pi_p}{\partial \alpha_p} = \frac{1}{\alpha_g - \alpha_p} - \frac{1}{3\alpha_g + \alpha_p - 1} \quad (45)$$

$$= \frac{2(\alpha_p + \alpha_g) - 1}{(\alpha_g - \alpha_p)(3\alpha_g + \alpha_p - 1)}. \quad (46)$$

On the other hand, if $\alpha_g > 2\alpha_p$, then

$$\frac{\partial \log \pi_p}{\partial \alpha_g} = \frac{2}{\alpha_g} - \frac{1}{\alpha_g - \alpha_p} - \frac{3}{3\alpha_g + \alpha_p - 1} \quad (47)$$

$$= \frac{2(\alpha_g - \alpha_p)(3\alpha_g + \alpha_p - 1) - \alpha_g(3\alpha_g + \alpha_p - 1) - 3\alpha_g(\alpha_g - \alpha_p)}{\alpha_g(\alpha_g - \alpha_p)(3\alpha_g + \alpha_p - 1)} \quad (48)$$

$$= \frac{-\alpha_g - 2\alpha_p\alpha_g - 2\alpha_p^2 + 2\alpha_p}{\alpha_g(\alpha_g - \alpha_p)(3\alpha_g + \alpha_p - 1)} \quad (49)$$

$$= \frac{(2\alpha_p - \alpha_g) - (2\alpha_p^2 + 2\alpha_p\alpha_g)}{\alpha_g(\alpha_g - \alpha_p)(3\alpha_g + \alpha_p - 1)} < 0 \ \text{(using Assumption 2 (i))} \quad (50)$$

The two relations (46) and (50) lead to

**Finding 2.** Suppose Assumptions 1 and 2 hold. In addition, suppose also that $\alpha_g > 2\alpha_p$. Then

(i) $\frac{\partial \pi_p}{\partial \alpha_g} < 0.$

(ii) $\frac{\partial \pi_p}{\partial \alpha_p} \gtreqless 0$ according as $2(\alpha_g + \alpha_p) - 1 \gtreqless 0.$

*Remark*: Note that if the *sufficient* condition for Assumption 2 (i) - (iii) to hold is satisfied, that is, (12) holds, then $\frac{\partial \pi_p}{\partial \alpha_p} > 0.$

The finding above shows that the private firm's profit is inversely related to $\alpha_g$ and positively related to $\alpha_p$. Private profit increases with a marginally higher weight in welfare and likewise decreases with a higher $\alpha_g$. This is consistent with the behavior of optimum prices from Finding 1.

**Behavior of the optimum welfare** From (15),

$$\frac{\partial \log W^{PU}}{\partial \alpha_g} = \frac{2}{\alpha_g} - \frac{3}{3\alpha_g + \alpha_p - 1} \quad (51)$$

$$= \frac{3\alpha_g + 2\alpha_p - 2}{\alpha_g(3\alpha_g + \alpha_p - 1)}. \quad (52)$$

Thus we obtain

**Finding 3.** Suppose Assumptions 1 and 2 hold.

(i) $\dfrac{\partial W^{PU}}{\partial \alpha_g} > 0 \text{ if } 3\alpha_g + 2\alpha_p > 2.$

(ii) $\dfrac{\partial W^{PU}}{\partial \alpha_g} < 0 \text{ if } 3\alpha_g + 2\alpha_p < 2.$

(iii) $\dfrac{\partial W^{PU}}{\partial \alpha_p} < 0.$

A detailed discussion of the welfare findings and comparison between the two cases is presented after Finding 6 in the next section.

### 3.2 Case 2: Privatization downstream

With the private monopoly downstream and public monopoly upstream, Proposition 2 summarizes the solutions to the maximization problem in Case 2. The downstream firm chooses output price $p_d$ to maximize profit, given upstream input price. The upstream public monopoly chooses input price to maximize welfare.

We first specify conditions that ensure non negative outcomes and also satisfy the second order conditions for maximization in Proposition 2. Assumption 3 (i) guarantees that the optimum satisfies $p_u - c_u > 0$, $p_d - p_u - c_d > 0$.

**Assumption 3.**

(i) $3\alpha_g > 1 + \alpha_p$.

(ii) $c_u + \theta_d c_d < 1$.

The solutions to the maximization problem in Case 2 are given in the following proposition.

**Proposition 2.** *Consider a bilateral monopoly with a public firm upstream and a private firm downstream. The public firm maximizes welfare and the private firm maximizes profit. Suppose Assumptions 1 and 3 hold. Then for every fixed $p_u$, the maximum value of $\pi_g(p_u, p_d)$ is obtained at a unique $p_d = p_d(p_u)$. The maximum value of $W(p_u, p_d(p_u))$ is obtained at a unique $p_u = p_u^0$. Further,*

$$p_u^0 = c_u + \frac{3\alpha_g - \alpha_p - 1}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d) \tag{53}$$

$$p_d^0 \equiv p_d(p_u^0) = 1 - \frac{\alpha_g}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d) \tag{54}$$

$$W^{PD} = W(p_u^0, p_d^0) = \frac{\alpha_g^2 (1 - c_u - \theta_d c_d)^2}{2(5\alpha_g - \alpha_p - 1)}. \tag{55}$$

**Proof**. Note that for fixed $p_u$,

$$\frac{\partial \pi_g}{\partial p_d} = (1 - p_d) - (p_d - p_u - \theta_d c_d) \text{ and } \frac{\partial^2 \pi_g}{\partial p_d^2} = -2. \tag{56}$$

Solving for $\frac{\partial \pi_g}{\partial p_d} = 0$ yields the maximum. This yields,

$$(1 - p_d) - (p_d - p_u - \theta_d c_d) = 0. \tag{57}$$

Let $p_d = p_d(p_u)$ be the solution of (57). Then

$$\frac{\partial p_d}{\partial p_u} = \frac{1}{2}. \tag{58}$$

Also note that using (57), the maximum value of $\pi_g$ for fixed $p_u$ equals

$$\pi_g = (1 - p_d)(p_d - p_u - c_d) = (1 - p_d)^2.$$

Then, using (57),

$$\begin{aligned} W(p_u, p_d(p_u)) &= (1 - p_d)[\alpha_g(p_u - c_u) + \alpha_p(p_d - p_u - \theta_d c_d)] + \alpha_c \frac{(1 - p_d)^2}{2} && (59)\\ &= (1 - p_d)\alpha_g(p_u - c_u) + \alpha_p(1 - p_d)^2 + \alpha_c \frac{(1 - p_d)^2}{2}. && (60) \end{aligned}$$

We now maximize $W = W(p_u, p_d(p_u))$ with respect to $p_u$. Using (57) and (58),

$$\begin{aligned} \frac{\partial W}{\partial p_u} &= \alpha_g(1 - p_d) + \alpha_g(p_u - c_u)(-\frac{1}{2}) + (\alpha_p + \frac{\alpha_c}{2})2(1 - p_d)(-\frac{1}{2}) && (61)\\ \frac{\partial^2 W}{\partial p_u^2} &= -\frac{\alpha_g}{2} - \frac{\alpha_g}{2} + (\alpha_p + \frac{\alpha_c}{2})2(-\frac{1}{2})(-\frac{1}{2}) && (62)\\ &= \frac{1}{2}[-2\alpha_g + \alpha_p + \frac{\alpha_c}{2}] < 0 \text{ since } 1 + \alpha_p < 3\alpha_g. && (63) \end{aligned}$$

Now note that

$$\begin{aligned} \frac{\partial W}{\partial p_u} &= 0 && (64)\\ \Rightarrow \quad & \alpha_g(1 - p_d) - \alpha_g \frac{(p_u - c_u)}{2} - (1 - p_d)(\alpha_p + \frac{\alpha_c}{2}) = 0 && (65)\\ \Rightarrow \quad & (1 - p_d)\left[\alpha_g - \frac{(2\alpha_p + \alpha_c)}{2}\right] = \frac{\alpha_g(p_u - c_u)}{2} && (66)\\ \Rightarrow \quad & (1 - p_d)[2\alpha_g - 2\alpha_p - \alpha_c] = \alpha_g(p_u - c_u) && (67)\\ \Rightarrow \quad & (1 - p_d)[2\alpha_g - 2\alpha_p - (1 - \alpha_g - \alpha_p)] = \alpha_g(p_u - c_u) \text{ using } \alpha_g + \alpha_p + \alpha_c = 1 && (68)\\ \Rightarrow \quad & (1 - p_d)[3\alpha_g - \alpha_p - 1] = \alpha_g(p_u - c_u). && (69) \end{aligned}$$

Using this, and (60), the maximized $W$ is given by

$$\begin{aligned} W &= (1 - p_d)[3\alpha_g - \alpha_p - 1](1 - p_d) + \alpha_p(1 - p_d)^2 + \frac{\alpha_c}{2}(1 - p_d)^2 && (70)\\ &= \frac{(1 - p_d)^2}{2}[6\alpha_g - 2 + \alpha_c] && (71)\\ &= \frac{(1 - p_d)^2}{2}[6\alpha_g - 2 + 1 - \alpha_g - \alpha_p] && (72)\\ &= \frac{(1 - p_d)^2}{2}[5\alpha_g - \alpha_p - 1]. && (73) \end{aligned}$$

It remains to find $p_d$. Using the value of $p_u$ from (57) in (69),

$$\begin{aligned} (1 - p_d)(3\alpha_g - \alpha_p - 1) &= \alpha_g(2p_d - 1 - \theta_d c_d - c_u) && (74)\\ &\Rightarrow p_d\,[(3\alpha_g - \alpha_p - 1) + 2\alpha_g] = (3\alpha_g - \alpha_p - 1) + \alpha_g(1 + c_u + \theta_d c_d) && (75)\\ &\Rightarrow p_d\,[5\alpha_g - \alpha_p - 1] = (4\alpha_g - \alpha_p - 1) + \alpha_g(c_u + \theta_d c_d) && (76)\\ &\Rightarrow 1 - p_d = \frac{\alpha_g}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d). && (77) \end{aligned}$$

Hence

$$
\begin{aligned}
W^{PD} &= \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2(5\alpha_g - \alpha_p - 1) && (78)\\
&= \frac{\alpha_g^2(1 - c_u - \theta_d c_d)^2}{2(5\alpha_g - \alpha_p - 1)} && (79)
\end{aligned}
$$

The value of $p_u$ is then easily obtained from (69) as

$$
\begin{aligned}
(1 - p_d)[3\alpha_g - \alpha_p - 1] &= \alpha_g(p_u - c_u) && (80)\\
\Leftrightarrow \frac{\alpha_g(3\alpha_g - \alpha_p - 1)}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d) &= \alpha_g(p_u - c_u) && (81)\\
\Leftrightarrow p_u - c_u &= \frac{3\alpha_g - \alpha_p - 1}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d) && (82)
\end{aligned}
$$

Finally, it is easily checked that (i) $p_u - c_u > 0$ if $3\alpha_g > 1 + \alpha_p$ and (ii) $p_d < 1$ if $5\alpha_g > 1 + \alpha_p$. Moreover,

$$p_u - c_u < 1 \Leftrightarrow (3\alpha_g - \alpha_p - 1)(1 - c_u - \theta_d c_d) < 5\alpha_g - \alpha_p - 1 \quad (83)$$

which holds if Assumption 3 holds. This completes the proof of the Proposition. ▲.

**Behavior of the optimum prices.** With the public firm upstream, input price is positively related to $\alpha_g$. This contrasts with Finding 1(i) where the reverse is true with the private firm upstream. The downstream price is positively related to $\alpha_g$ in both cases. So when the profit of the public firm is marginally more important in the welfare function, the public firm increases its price, regardless of whether it is the upstream or the downstream firm. When the public firm is upstream and increases its price, the price of the downstream private firm has to increase to pay the higher input costs. However, when the private firm is upstream, its profit maximizing price is actually lower, to compensate for the higher downstream price.

It is interesting to compare the above intuition to what happens when the private firm's profit is marginally more important in the welfare function. With the public firm upstream, input price is inversely related to $\alpha_p$. This contrasts with Finding 1(i) where the reverse is true with the private firm upstream. The downstream price is inversely (positively) related to $\alpha_p$ with the public firm upstream (downstream). So when the profit of the private firm is marginally more important, the public firm decreases its input price as the upstream firm, thus enabling the downstream firm to also lower its price in response to lower input costs. When the private firm is upstream, its price increases in a bid for higher profit and the downstream price of the public firm has to increase to pay the higher input costs, because the public firm is not subsidized, following Assumptions 2 and 3. Finding 4 summarizes the behavior of optimum prices for Case 2.

**Finding 4.** Under Assumptions 1 and 3,

(i) $\dfrac{\partial p_u^0}{\partial \alpha_g} > 0$ and $\dfrac{\partial p_u^0}{\partial \alpha_p} < 0$.

(ii) $\dfrac{\partial p_d^0}{\partial \alpha_g} > 0$ and $\dfrac{\partial p_d^0}{\partial \alpha_p} < 0$.

**Behavior of the optimum profit of the private firm downstream.** Observing (60) and using (54), the optimized profit $\pi_p(p_u^0, p_d^0)$ is given by

$$\pi_p \equiv \pi_p(p_u^0, p_d^0) = (1 - p_d^0)^2 = \left[\frac{\alpha_g}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d)\right]^2.$$

We then easily obtain the following.

**Finding 5.** Under Assumptions 1 and 3,

(i) $\dfrac{\partial \pi_p}{\partial \alpha_g} < 0.$

(ii) $\dfrac{\partial \pi_p}{\partial \alpha_p} > 0.$

Comparing Finding 5 with Finding 2, we note that if the share of public firm's profit in welfare is marginally higher, then as expected, the private firm's profit falls, regardless of whether the private firm is upstream or downstream. For exactly the same reasons, if the share of private firm's profit in welfare is marginally higher, the reverse is true.

**Behavior of the optimum welfare.** Note that

$$\frac{\partial \log W^{PD}}{\partial \alpha_g} = \frac{2}{\alpha_g} - \frac{5}{5\alpha_g - \alpha_p - 1} \tag{84}$$

$$= \frac{5\alpha_g - 2\alpha_p - 2}{\alpha_g(5\alpha_g - \alpha_p - 1)}. \tag{85}$$

As a consequence, we obtain,

**Finding 6.** Under Assumptions 1 and 3,

(i) $\dfrac{\partial W^{PD}}{\partial \alpha_g} < 0 \ \text{ if } \ \dfrac{5}{2}\alpha_g < 1 + \alpha_p < 3\alpha_g.$

(ii) $\dfrac{\partial W^{PD}}{\partial \alpha_g} > 0 \ \text{ if } \ \dfrac{5}{2}\alpha_g > 1 + \alpha_p.$

(iii) $\dfrac{\partial W^{PD}}{\partial \alpha_p} > 0.$

Comparing Finding 3 with Finding 6, we note that if the share of public firm's profit in welfare is marginally higher, welfare may increase or decrease in either case, privatization upstream or downstream. This will depend on the relative shares of public and private profit, in welfare. Typically we find that welfare will increase (decrease) if the weight of the public firm's profit is relatively large (small). This is intuitively reasonable because it shows that the effect of higher $\alpha_g$ on welfare, is driven by the relative weight of the public firm's profit in the welfare function itself. On the other hand, if the share of private firm's profit in welfare, $\alpha_p$ is marginally higher, welfare decreases (increases) when the private firm is upstream (downstream). Using the behavior of the optimum profit and prices from above, we can see that with privatization downstream, the higher private profit, with a marginal increase in $\alpha_p$, increases welfare in this case. However, when the private firm is upstream, and both prices increase, then a marginal increase in $\alpha_p$ will consequently lower welfare, even though private profit has increased. We speculate that this is because higher private profit is not sufficient to compensate for the negative effects on welfare from a higher downstream price. In the next section we discuss the welfare comparisons between the two cases in further detail.

### 3.3 Welfare comparison

The question that remains to be answered is, which regime–an upstream private firm (and public firm downstream) or a downstream private firm (and public firm upstream), will have higher welfare? To begin the discussion, Proposition 3 summarizes the welfare findings from Propositions 1 and 2.

**Proposition 3.** *Suppose Assumptions 1, 2 and 3 hold. Then*

$$W^{PU} \gtreqless W^{PD} \textit{iff } \frac{(1-\theta_u c_u - c_d)^2}{4(3\alpha_g+\alpha_p-1)} \gtreqless \frac{(1-c_u-\theta_d c_d)^2}{(5\alpha_g-\alpha_p-1)}.$$

Clearly, casual observation of the proposition, does not immediately provide an answer. Suppose $\beta(1-c_u-\theta_d c_d)^2 = (1-\theta_u c_u - c_d)^2$, for some $\beta$. Then it is easily checked that

$$W^{PU} \gtreqless W^{PD} \tag{86}$$

$$\Leftrightarrow \frac{(5\alpha_g-\alpha_p-1)\beta}{4(3\alpha_g+\alpha_p-1)} \gtreqless 1 \tag{87}$$

$$\Leftrightarrow 4-\beta \gtreqless \alpha_g(12-5\beta)+(4+\beta)\alpha_p. \tag{88}$$

Substituting $\beta = 1$, we get parts (i) and (ii) of the Corollary below.

What these two results show is that even if the marginal cost of production in the vertical stream is the same in both cases, welfare is not necessarily identical. Further, welfare with privatization upstream is less than welfare with privatization downstream if $\alpha_c$ is not very large. In other words, if the marginal cost of production in the same and the weight of consumer surplus in the welfare is not very large, then welfare with privatization downstream is greater than welfare with privatization upstream.

To complete the analysis, if the marginal cost of production in the vertical stream is not equal, we find that the welfare results from privatization upstream or downstream may go either way. See part (iii) of Corollary below.

**Corollary 1.** *Suppose Assumptions 1, 2 and 3 hold. Then*

*(i) If* $c_u(1-\theta_u) = c_d(1-\theta_d)$ *and* $7\alpha_g + 5\alpha_p > 3$*, then* $W^{PU} < W^{PD}$.

*(ii) If* $c_u(1-\theta_u) = c_d(1-\theta_d)$ *and* $7\alpha_g + 5\alpha_p = 3$*, then* $W^{PU} = W^{PD}$.

*(iii) If* $\beta(1-c_u-\theta_d c_d)^2 = (1-\theta_u c_u - c_d)^2$*, then*

$$W^{PU} \gtreqless W^{PD} \textit{ iff } 4-\beta \gtreqless \alpha_g(12-5\beta)+(4+\beta)\alpha_p.$$

To complement the results in (i) and (ii), we make the particular point that from (iii) above, if $\beta = 12/5$ then $W^{PU} > W^{PD}$ if $\alpha_p < 1/4$. In other words, if $\alpha_p$ is relatively small, then privatization upstream may yield higher welfare than privatization downstream, under some conditions.

## 4 Conclusions

The paper analyzes privatization in a bilateral monopoly to study the welfare effects from privatizing either upstream or downstream. We find the optimal input and output prices, private profit and public welfare and analyze their behavior in each framework. The welfare implication from privatization upstream or downstream is that even if the productive efficiency benefits are equal, the implications from a welfare perspective are not necessarily so. We further show that there are conditions under which welfare may be higher or lower with privatization upstream or downstream. Hence specific policies directed at privatizing either upstream or downstream may have theoretical support.

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