Martingale Problems

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Outline

1 Introduction
   • Definition
   • Well-posedness

2 Examples-Finite Dimensions
   • Brownian Motion
   • Poisson Process
   • Diffusions
   • Markov Jump Processes

3 Examples - Infinite Dimensions
   • Hilbert Space Valued Diffusion
   • Measure valued processes
Definition 1.1

An $E$-valued measurable process $(X_t)_{t \geq 0}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a solution of the martingale problem for $(A, \mu)$ with respect to a filtration $(\mathcal{G}_t)_{t \geq 0}$ if

1. $\mathcal{L}(X_0) = \mu$
2. for every $f \in D(A)$

\[ M^f_t = f(X_t) - \int_0^t Af(X_s) \, ds \]

is a $(\mathcal{G}_t)_{t \geq 0}$-martingale.
General Setup

- **State space** - $E$ - a complete, separable metric space
  - $M(E)$ - real valued, measurable functions on $E$
  - $B(E)$ - real valued, bounded, measurable functions on $E$
  - $C(E)$ - real valued, continuous functions on $E$
  - $C_b(E)$ - real valued, bounded, continuous functions on $E$
- **operator** $A$ on $M(E)$ with domain $D(A)$
  - $\mathcal{B}(E)$ - Borel $\sigma$-field on $E$
  - $\mathcal{P}(E)$ - space of probability measures on $(E, \mathcal{B}(E))$
- **Initial measure** $\mu \in \mathcal{P}(E)$
- For any process $(X_t)_{t \geq 0}$, $(\mathcal{F}_t^X)_{t \geq 0}$ will denote its natural filtration. i.e.
  \[ \mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t) \]

In Definition 1.1 if $(\mathcal{G}_t)_{t \geq 0} = (\mathcal{F}_t^X)_{t \geq 0}$, the $\sigma$-fields are dropped from the statement.
Solution of a martingale problem is defined only in a weak sense.

**Definition 1.2**

*Uniqueness* holds for the martingale problem for \((A, \mu)\) if any two solutions of the martingale problem have the same distributions.

**Definition 1.3**

The martingale problem for \((A, \mu)\) is *well-posed* if

1. there exists a solution \(X\) of the martingale problem for \((A, \mu)\)
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Example 1 - Brownian Motion

Let $B$ be a Standard Brownian Motion.

Then $M_t^1 = B_t$ and $M_t^2 = B_t^2 - t$ are martingales.

Let

$$E = \mathbb{R}, D(A) = \{f_1, f_2\}$$

$$f_1(x) = x, f_2(x) = x^2$$

$$Af_1(x) \equiv 0, Af_2(x) \equiv 1$$

Then for $i = 1, 2$

$$M_t^i = f_i(B_t) - \int_0^t Af_i(B_s)ds.$$

∴ $(B_t)_{t \geq 0}$ is a solution of the martingale problem for $(A, \delta_0)$
Conversely,
Let \( (X_t)_{t \geq 0} \) be a continuous solution of the martingale problem for \((A, \delta_0)\).
We have \( X_t \) and \( X_t^2 - t \) are martingales.
i.e. \( X_t \) is a continuous martingale with \( \langle X \rangle_t = t \).
Define \( g_s(x) = e^{isx} \) where \( i = \sqrt{-1} \). Note \( |g_s(x)| \leq 1 \).
Then \( g_s'(x) = isg_s(x), g_s''(x) = -s^2g_s(x) \)
By Ito’s formula
\[
\begin{align*}
dg_s(X_t) &= isg_s(X_t)dX_t - \frac{1}{2}s^2g_s(X_t)dt.
\end{align*}
\]
The stochastic integral is a martingale \( M_t \). Then for \( 0 \leq r < t \)
\[
\begin{align*}
e^{isX_t} &= e^{isX_r} + M_t - M_r - \frac{1}{2}s^2 \int_r^t e^{isX_u} du
\end{align*}
\]
Example 1 - Brownian Motion (Contd.)

Let \( A \in \mathcal{F}_r^X \). Multiplying by \( e^{-isX_r}I_A \)

\[
I_Ae^{is(X_t-X_r)} = I_A + e^{-isX_r}I_A(M_t-M_r) - \frac{1}{2}s^2 \int_r^t I_Ae^{is(X_u-X_r)} du
\]

Taking expectations (of conditional expectations)

\[
\mathbb{E} \left[ I_Ae^{is(X_t-X_r)} \right] = \mathbb{P}(A) + 0 - \frac{1}{2}s^2 \int_r^t \mathbb{E} \left[ I_Ae^{is(X_u-X_r)} \right] du
\]

Let \( h(t) = \mathbb{E} \left[ I_Ae^{is(X_t-X_r)} \right] \). Then

\[
h(t) = \mathbb{P}(A) - \frac{1}{2}s^2 \int_r^t h(u)du
\]

\[
h'(t) = -\frac{1}{2}s^2 h(t) \text{ with } h(r) = \mathbb{P}(A)
\]

\[
\mathbb{E} \left[ I_Ae^{is(X_t-X_r)} \right] = h(t) = \mathbb{P}(A)e^{-\frac{1}{2}s^2(t-r)}.
\]
Example 1 - Brownian Motion (Contd.)

Since this holds for all $A \in \mathcal{F}_r^X$, we get

$$\mathbb{E}\left[e^{is(X_t-X_r)}|\mathcal{F}_r^X\right] = e^{-\frac{1}{2}s^2(t-r)} \text{ a.s.}$$

This implies

$$(X_t - X_r) \bigcup \mathcal{F}_r^X \text{ independent increments}$$

$$(X_t - X_r) \sim N(0, t-r) \text{ Stationary, Gaussian}$$

Thus $X_t$ is a Brownian motion.

This is Levy’s Characterization of Brownian Motion

- The martingale problem for $(A, \delta_0)$ is well-posed in the class of continuous processes
Levy’s Characterization Theorem

Let \((X_t)_{t \geq 0}\) be a continuous \(\mathbb{R}^d\) valued process, with
\((X_t = (X_t^{(1)}, \ldots, X_t^{(d)}))\), such that for every \(1 \leq k, j \leq d\)

1. \(M_t^{(k)} = X_t^{(k)} - X_0^{(k)}\) is a continuous local martingale
2. \(\langle M^{(k)}, M^{(j)} \rangle_t = \delta_{kj} t\) i.e. \(M_t^{(k)} M_t^{(j)} - \delta_{kj} t\) is a continuous local martingale

Then \((X_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian Motion.
Levy’s Characterization Theorem

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Then \((X_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian Motion.

(local martingale \((M_t):\) \(\exists\) a sequence of stop-times \(\tau_n \uparrow \infty\) such that for every \(n \geq 1\), the stopped process \((M^n_t)\) defined by 
\[ M^n_t = M_{t \wedge \tau_n} \]

is a martingale.)
Example 2 - Compensated Poisson Process

- Let \((N_t)_{t \geq 0}\) be a Poisson Process with intensity 1
- Define \(\tilde{N}_t = N_t - t\), (compensated Poisson process)
- Using independent increment property of \(N\), it follows that \(\tilde{N}_t\) and \(\tilde{N}_t^2 - t\) are martingales
- \((\tilde{N}_t)_{t \geq 0}\) is also a solution of the martingale problem for \((A, \delta_0)\) of Example 1

The martingale problem for \((A, \delta_0)\) is not well-posed though uniqueness holds in the class of continuous solutions.
Example 3 - Diffusions

Let $b(x) = (b_i(x))_{1 \leq i \leq d}$, $\sigma(x) = ((\sigma_{ij}(x)))_{1 \leq i,j \leq d}$ be measurable functions, and 
$$(W_t)_{t \geq 0} = (W_t^{(1)}, \ldots, W_t^{(d)})_{t \geq 0}$$
be a $d$-dimensional Standard Brownian Motion.

Suppose that (the $d$-dimensional process) $X$ is a solution of the Stochastic Differential Equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

$$dX_t^{(i)} = b_i(X_t)dt + \sum_{j=1}^{d} \sigma_{ij}(X_t)dW_t^{(j)} \quad 1 \leq i \leq d$$

$$X_t^{(i)} = X_0^{(i)} + \int_0^t b_i(X_s)ds + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(X_s)dW_s^{(j)} \quad 1 \leq i \leq d$$
Then, by Ito’s formula, for $f \in C_b^2(\mathbb{R}^d)$

$$df(X_t) = \sum_{i=1}^{d} \partial_i f(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t$$
Then, by Ito's formula, for \( f \in C^2_b(\mathbb{R}^d) \)

\[
    df(X_t) = \sum_{i=1}^{d} \partial_i f(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t
\]
Then, by Ito’s formula, for \( f \in C^2_b(\mathbb{R}^d) \)

\[
\begin{align*}
    df(X_t) &= \sum_{i=1}^{d} \partial_i f(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t \\
    &= \sum_{i=1}^{d} \partial_i f(X_t) b_i(X_t) dt + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) (\sigma \sigma^T)_{ij}(X_t) dt \\
    &\quad + \sum_{i=1}^{d} \partial_i f(X_t) \sigma_{ij}(X_t) dW_t^{(j)}.
\end{align*}
\]
Then, by Ito’s formula, for $f \in C^2_b(\mathbb{R}^d)$

$$
\frac{df(X_t)}{dt} = \sum_{i=1}^{d} \partial_i f(X_t) b_i(X_t) dt + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) \langle X^{(i)}, X^{(j)} \rangle_t + \sum_{i=1}^{d} \partial_i f(X_t) \sigma_{ij}(X_t) dW_t^{(j)}.
$$

Let $Af(x) = \sum_{i=1}^{d} \partial_i f(x) b_i(x) + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(x) (\sigma \sigma^T)_{ij}(x)$
Thus for all $f \in D(A) = C^2_b(\mathbb{R}^d)$\)

$$f(X_t) - \int_0^t Af(X_s)ds$$

is a martingale.

Or, $X_t$ is a solution of the $(A, \mu)$ martingale problem where $\mu = \mathcal{L}(X_0)$. 

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**Converse!!!**

**Stroock-Varadhan Theory of Martingale Problems**
Example 3 - Diffusions (Contd.)

Thus for all $f \in D(A) = C^2_b(\mathbb{R}^d)$

$$f(X_t) - \int_0^t Af(X_s)ds$$

is a martingale.

Or, $X_t$ is a solution of the $(A, \mu)$ martingale problem where $\mu = \mathcal{L}(X_0)$.

Converse!!!

**Stroock-Varadhan Theory of Martingale Problems**
Example 3: Diffusions - Martingale Characterization

- $E = \mathbb{R}^d; D(A) = C_b^2(\mathbb{R}^d)$
- $b(x) = (b_i(x))_{1 \leq i \leq d}, \sigma(x) = ((\sigma_{ij}(x)))_{1 \leq i \leq d, 1 \leq j \leq d}$ be measurable functions
- $Af(x) = \sum_{i=1}^{d} \partial_i f(x)b_i(x) + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(x)(\sigma\sigma^T)_{ij}(x)$

Theorem 1

Let $(X_t)_{t \geq 0}$ (defined on some $(\Omega, \mathcal{F}, \mathbb{P})$) be a continuous $\mathbb{R}^d$ valued solution of the martingale problem for $(A, \mu)$. Then $\exists$ a d-dimensional Brownian motion $(W_t)_{t \geq 0}$, defined possibly on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $(X_t)_{t \geq 0}$ solves the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad \mathcal{L}(X_0) = \mu. \quad (1)$$
Example 3: Diffusions - Martingale Characterization

Proof. Let \( a(x) = (\sigma \sigma^T)(x) \).
If \( f_k(x) = x^k, g_{kl}(x) = x^k x^l \in D(A) \), then

\[
M_t^k = X_t^k - \int_0^t b_k(X_s) ds
\]

(2)
is a martingale, since \( \partial_i f_k \equiv \delta_{ik}, \partial_{ij} f_k \equiv 0 \).
Example 3: Diffusions - Martingale Characterization

Proof. Let $a(x) = (\sigma \sigma^T)(x)$.
If $f_k(x) = x^k, g_{kl}(x) = x^k x^l \in D(A)$, then

$$M^k_t = X^k_t - \int_0^t b_k(X_s) ds$$

(2)

is a martingale, since $\partial_i f_k \equiv \delta_{ik}, \partial_{ij} f_k \equiv 0$.
Using the functions $g_{kl}$ and their partial derivatives we can write

$$M^k_t M^l_t - \int_0^t a_{kl}(X_s) ds$$

as sum of martingales

$$g_{kl}(X_t) - \int_0^t A g_{kl}(X_s) ds, \quad \int_0^t Z^k_s dM^l_s, \quad \int_0^t Z^l_s dM^k_s$$

where $Z^j_s = \int_0^s b_j(X_r) dr, j = k, l$. 
Example 3: Diffusions - Martingale Characterization

Thus

$$\langle M^k, M^l \rangle_t = \int_0^t a_{kl}(X_s) ds$$  (3)

However, $f_k, g_{kl} \not\in D(A)$.

Define $f_{k,n}, g_{kl,n} \in D(A)$:

Let $B(0, n) = \text{the ball of radius } n \text{ with center } 0$

$$f_{k,n}(x) = x^k, g_{kl,n}(x) = x^k x^l \text{ on } B(0, n)$$

$$f_{k,n}(x) = g_{kl,n}(x) = 0 \text{ on } B(0, n + 1)^c$$

Then using stop-times

$$\tau_n = \inf\{t \geq 0 : X_t \not\in B(0, n)\}$$

we see that (2) is a local martingale and such that (3) holds.
Example 3: Diffusions - Martingale Characterization

Now, if $\sigma$ is invertible, define

$$W_t = \int_0^t \sigma^{-1}(X_s) dM_s, \text{ or}$$

$$W_t^i = \sum_{k=1}^{d} \int_0^t \sigma_{ik}^{-1}(X_s) dM^k_s, \quad 1 \leq i \leq d$$

Then, (2) $\implies (W_t^k)_{\{t \geq 0\}}$ is a local martingale; (3) $\implies$

$$\langle W^i, W^j \rangle_t = \sum_{k,l=1}^{d} \left\langle \int_0^t \sigma_{ik}^{-1}(X_s) dM^k_s, \int_0^t \sigma_{jl}^{-1}(X_s) dM^l_s \right\rangle$$

$$\quad = \sum_{k,l=1}^{d} \int_0^t \left( \sigma_{ik}^{-1} a_{kl}(\sigma^T)^{-1}_{lj} \right) (X_s) ds$$

$$\quad = \delta_{ii} t$$
Example 3: Diffusions - Martingale Characterization

Thus Levy’s Characterization theorem implies that $W$ is a $d$-dimensional Brownian motion. Finally,

$$
\int_0^t \sigma(X_s) dW_s = \int_0^t dM_s = X_t - \int_0^t b(X_s) ds
$$

and hence $X$ is a solution of the SDE (1).
Thus Levy’s Characterization theorem implies that $W$ is a $d$-dimensional Brownian motion.

Finally,

$$\int_0^t \sigma(X_s)dW_s = \int_0^t dM_s = X_t - \int_0^t b(X_s)ds$$

and hence $X$ is a solution of the SDE (1).

- When $\sigma$ is singular - it is possible that the space $\Omega$ may not be rich enough to hold a Brownian motion.
- Intuitively, we plug in another Brownian motion wherever $\sigma$ is degenerate.
Example 3: Diffusions - Martingale Characterization

Consider \((\Gamma, \mathcal{G}, \mathbb{Q})\) and a Brownian motion \((B_t)_{t \geq 0}\) defined on it.

Get \(d \times d\) (measurable) matrices \(\rho(x), \eta(x)\) satisfying:

- \(\rho \rho^T + \eta \eta^T = I_d\)
- \(\rho \eta = 0\)
- \((I_d - \sigma \rho)(I_d - \sigma \rho)^T = 0\)

Define on \((\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Gamma, \mathcal{G}, \mathbb{Q})\)

\[
W_t = \int_0^t \rho(X_s) dM_s + \int_0^t \eta(X_s) dB_s
\]

Then \(W\) is a Brownian motion on the extended space and (1) holds.
Example 4: Markov Jump Process

- Let $\mu(x, \Gamma)$ be a transition function on $E \times \mathcal{E}$ and $\lambda > 0$.
- Let $\{Y_0, Y_1, Y_2, \ldots\}$ be a Markov chain
  - $P(Y_0 \in \Gamma) = \nu(\Gamma)$
  - $P(Y_{k+1} \in \Gamma | Y_0, \ldots, Y_k) = \mu(Y_k, \Gamma)$
- Let $N$ be a Poisson process with intensity $\lambda$, independent of $Y$.
- Define $X$ by
  $$X_t = Y_{N_t}, \quad t \geq 0$$
  Then $X$ is a process which jumps at exponential times and the jump is dictated by the transition function $\mu(\cdot, \cdot)$.
- Define
  $$Pf(x) = \int_E f(y)\mu(x, dy)$$
Example 4: Markov Jump Process (Contd.)

Note for $F_1 \in \mathcal{F}_t^N$ and $F_2 \in \mathcal{F}_t^Y$

$$
\mathbb{E} \left[ f(Y_{k+N_t}) \mathbb{I}_{F_1 \cap F_2 \cap \{N_t = l\}} \right] = \mathbb{E} \left[ f(Y_{k+l}) \mathbb{I}_{F_1 \cap F_2 \cap \{N_t = l\}} \right]
= \mathbb{E} \left[ P^k f(Y_l) \mathbb{I}_{F_2} \right] \mathbb{P}(F_1 \cap \{N_t = l\})
= \mathbb{E} \left[ f(X_t) \mathbb{I}_{F_1 \cap F_2 \cap \{N_t = l\}} \right]
$$

- $F_1 \cap F_2 \cap \{N_t = l\}$ generate $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{F}_t^X$

- Thus

$$
\mathbb{E} \left[ f(Y_{k+N_t})|\mathcal{F}_t \right] = P^k f(X_t) \text{ a.s.}
$$

- Using, independent increments of $N$

$$
\mathbb{E} \left[ f(X_{t+s})|\mathcal{F}_t \right] = \mathbb{E} \left[ f(Y_{N_{t+s}-N_t+N_t})|\mathcal{F}_t \right]
= \sum_{k=0}^{\infty} e^{-\lambda s} \frac{\lambda^k s^k}{k!} P^k f(X_t)
$$
Finally

\[ T_t = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} P^k \]

defines a one parameter operator semigroup

**Generator** \( A \): \( T_t = e^{tA} \)

\[ A = \lambda(P - I) \]

\[ Af(x) = \lambda \int_E (f(y) - f(x)) \mu(x, dy) \]

\( X \) is a solution of the martingale problem for \((A, \nu)\)
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Example 5: Hilbert Space Valued Diffusion

Let $E = H$, a real, separable Hilbert space, with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. 

$L_2(H, H)$ - the space of Hilbert Schmidt operators on $H$ 
i.e. $\Sigma \in L_2(H, H)$ iff $\| \Sigma \|_{HS} = \sum_i (\Sigma \phi_i, \Sigma \phi_i) < \infty$, Hilbert Schmidt norm 

Let $\sigma : H \to L_2(H, H)$, $b : H \to H$ be measurable

\[ \| \sigma(h) \|_{HS} \leq K \]
\[ \| b(h) \| \leq K \]
\[ \| \sigma(h_1) - \sigma(h_2) \|_{HS} \leq K \| h_1 - h_2 \| \]
\[ \| b(h_1) - b(h_2) \| \leq K \| h_1 - h_2 \| \]

for all $h, h_1, h_2 \in H$. 
Example 5: Hilbert Space Valued Diffusion (Contd.)

- Fix a Complete OrthoNormal System \( \{ \phi_i : i \geq 1 \} \) in \( H \)
- Let \( P_n : H \to \mathbb{R}^n \) be defined by
  \[
P_n(h) = ((h, \phi_1), \ldots, (h, \phi_n)).
  \]
- \( D(A) = \{ f \circ P_n : f \in C^2_c(\mathbb{R}^n), n \geq 1 \} \),
  \[
  [A(f \circ P_n)](h) = \frac{1}{2} \sum_{i,j=1}^{n} (\sigma^*(h)\phi_i, \sigma^*(h)\phi_j) \partial_{ij} f \circ P_n(h) \\
  + \sum_{i=1}^{n} (b(h), \phi_i) \partial_i f \circ P_n(h)
  \]

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Example 5: Hilbert Space Valued Diffusion (Contd.)

- The martingale problem for \((A, \mu)\) is well-posed.
- The unique solution \(X\) is continuous a.s.
- \(\exists\) Cylindrical Brownian motion \(B\) on some \((\Omega, \mathcal{F}, \mathbb{P})\)
  - \((B_t, h)\) is a 1-dimensional Brownian Motion for all \(h \in H\)
  - \(\mathbb{E}[(B_t, h_1)(B_t, h_2)] = (h_1, h_2)\) for all \(h_1, h_2 \in H\)
- It is a Hilbert space valued diffusion. i.e.

\[
  dX_t = \sigma(X_t)dB_t + b(X_t)dt
\]

for some Cylindrical Brownian motion \(B\)
Example 6: Branching Brownian Motion

**Initial Configuration** Individuals in the population are scattered in $\mathbb{R}^d$.

**Spatial Motion** Each individual, during its lifetime, moves in $\mathbb{R}^d$ according to a Brownian motion, independently of all other particles.

**Branching rate, $\alpha$** Each individual has an exponentially distributed lifetime $\alpha$.

**Branching mechanism, $\Phi$** When the individual dies, it leaves behind at the same location a random number of offsprings with probability generating function

$$
\Phi(s) = \sum_{l=0}^{\infty} p_l s^l
$$
Example 6: Branching Brownian Motion (Contd.)

Let $X$ denote such a process

- state space $E' = \{(k, x_1, \ldots, x_k) : k = 0, 1, 2, \ldots, x_i \in \mathbb{R}^d\}$.
- Consider functions $f(k, x_1, \ldots, x_k) = \prod_{i=1}^{k} g(x_i)$ on $E$
- Generator of Brownian motion - $L_1 = \frac{1}{2} \Delta$
- Generator for the Branching process
  $L_2 h(k) = \sum_{l=0}^{\infty} \alpha k p_l (h(k - 1 + l) - h(k))$
- In the absence of branching

$$A_1 \left( \prod_{i=1}^{k} g(x_i) \right) = \sum_{j=1}^{k} L_1 g(x_j) \prod_{i \neq j} g(x_i)$$

so that $f(X_t) - \int_{0}^{t} A_1 f(X_s) ds$ is a martingale
Example 6: Branching Brownian Motion (Contd.)

- In presence of branching but no motion

\[ A_2 \left( \prod_{i=1}^{k} g(x_i) \right) = \sum_{j=1}^{k} \alpha (\Phi(g(x_j)) - g(x_j)) \prod_{i \neq j} g(x_i) \]

- Independence of branching and motion suggest that the “martingale problem operator” for \( X \) should be

\[ A = A_1 + A_2 \]
Example 6: Branching Brownian Motion (Contd.)

- In presence of branching but no motion

\[ A_2 \left( \prod_{i=1}^{k} g(x_i) \right) = \sum_{j=1}^{k} \alpha (\Phi(g(x_j)) - g(x_j)) \prod_{i \neq j} g(x_i) \]

- Independence of branching and motion suggest that the “martingale problem operator” for $X$ should be

\[ A = A_1 + A_2 \]

- $E'$ is cumbersome to work with
- order of particles is not important
Example 6: Branching Brownian Motion (Contd.)

- \( E = \left\{ \sum_{i=1}^{k} \delta_{x_i} : k = 0, 1, 2, \ldots; x_i \in \mathbb{R}^d \right\} \)
- \( E \subset \mathcal{M}(\mathbb{R}^d) \), space of positive finite measures on \( \mathbb{R}^d \)
- For \( \mu = \sum_{i=1}^{k} \delta_{x_i} \)

\[
\prod_{i=1}^{k} g(x_i) = e^{\langle \log g, \mu \rangle}
\]

- \( \mathcal{A} e^{\langle \log g, \mu \rangle} = e^{\langle \log g, \mu \rangle} \left\langle \frac{L_1 g + \alpha (\Phi(g) - g)}{g}, \mu \right\rangle \)

- \( \xi_t = \sum \delta_{X_t} \) is a solution of the martingale problem for \( \mathcal{A} \)