NONPARAMETRIC STATISTICS ON MANIFOLDS WITH APPLICATIONS TO SHAPE SPACES

Abhishek Bhattacharya
Department of Mathematics, University of Arizona, Tucson
Advisor: Prof. R. Bhattacharya

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Shapes of k-ads

- k points are picked from an object in 2D or 3D usually with expert help called k-ad or configuration of k points.
- In general, each observation $\mathbf{x} = (x_1, \ldots, x_k)$ consists of $k > m$ points in m-dimension (not all same).
- Shape of a k-ad is its orbit under a group G of transformations.

Different Notions of Shapes of k-ads

- Kendall's (Direct) Similarity Shape Space Σ_m^k. Group G generated by translations, scaling and rotations.

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Different Notions of Shapes of k-ads

- Kendall's (Direct) Similarity Shape Space Σ_m^k. Group G generated by translations, scaling and rotations.
- Reflection Similarity Shape Space $R\Sigma_m^k$. Remove the effects of translations, scaling and all orthogonal transformations.
- Similarity shape analysis finds applications in morphometrics - classification of biological species based on their shapes, medical diagnostics - disease detection based on change in shape of an organ due to disease or deformation, evolution studies - studying the change in shape of an organ or organism with time, age etc.

Different Shapes of k-ads (cont.)

- Affine Shape Space $A\Sigma_m^k$. Group G consists of all affine transformations.
- Applications in protein matching, in scene recognition: to reconstruct a larger image from partial views in a number of ariel images of that scene.

Different Shapes of k-ads (cont.)

- Affine Shape Space $A\Sigma_m^k$. Group G consists of all affine transformations.
- Applications in protein matching, in scene recognition: to reconstruct a larger image from partial views in a number of ariel images of that scene.
- Projective Shape Space $P\Sigma_m^k$. Landmarks of the k-ad viewed in $\mathbb{R}P^m$-the set of all lines through origin in $\mathbb{R}^{m+1}$. The projective shape of a k-ad is invariant under all projective transformations.
- Applications in image analysis, robotics - for robots to visually recognize a scene.

Relation between different notions of shapes

- When images/photos obtained through a central projection, like a pinhole camera, projective shape analysis is useful.
- For images takes from a great distance, the rays from the object almost parallel to the camera plane. Then affine shape analysis appropriate.
- Further if the rays are perpendicular to the camera plane, similary shapes can be used.

Fréchet Function

- Let (M, ρ) be a metric space.
- Let Q be a probability distribution on M.
- Fréchet function of Q is defined as

$$F(p) = \int_M \rho^\alpha(p, x) Q(dx), \ p \in M$$

for given $\alpha \geq 1$.

Fréchet Mean and Variation

Definition

Suppose $F(p) < \infty$ for some $p \in M$. The Fréchet mean set of Q is the set of all minimizers of F, denoted as C_Q. A unique minimizer (if exists) is called the Fréchet mean of Q, say μ_F.

Definition

The infimum of F on M (if finite) is called the Fréchet variation of Q.

Existence of Fréchet Mean Set C_Q and Variation V

Proposition

Suppose every closed and bounded subset of M is compact. If $F(p) < \infty$ for some $p \in M$, then C_Q is nonempty and compact.

In view of Proposition 3.1, the Frechet variation V is attained and it is the value of the Fréchet function on C_Q.

Sample Fréchet Mean and Variation

Definition

Given an iid sample $X_1, \dots, X_n$ from Q, the sample Fréchet mean μ_{F_n} is a measurable selection from the Fréchet mean set of the empirical distribution Q_n.

Definition

The Fréchet variation of Q_n is called the sample Fréchet variation.

Consistency of the Sample Estimates

Suppose every closed and bounded subset of M is compact and F is finite on M.

Proposition

The sample Fréchet variation V_n is a strongly consistent estimator of the Fréchet variation V of Q.

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Proposition

If Q has a unique Fréchet mean μ_F, then the sample Fréchet mean μ_{F_n} is a strongly consistent estimator of μ_F.

Asymptotic Distribution of Sample Fréchet Mean

- Let M be a differentiable manifold of dimension d.
- Let ρ be a distance metrizing the topology of M.
- Suppose the following assumptions hold:

Assumptions

- A1 Q has support in a single coordinate patch, (U, ϕ). Let $\tilde{X}_j = \phi(X_j)$, $j = 1, \dots, n$.
- A2 Fréchet mean μ_F of Q is unique.
- A3 For all $x, y \mapsto h(x, y) = \rho^\alpha\left(\phi^{-1}(x), \phi^{-1}(y)\right)$ is twice continuously differentiable in a neighborhood of $\phi(\mu_F) = \mu$.
- A4 $\mathrm{E}\{\mathrm{D}_r h(\tilde{X}_1, \mu)\}^2 < \infty \; \forall r = 1, \dots, d$.
- A5 $\mathrm{E}\{\sup_{|u-v| \le \epsilon} |\mathrm{D}_s \mathrm{D}_r h(\tilde{X}_1, v) - \mathrm{D}_s \mathrm{D}_r h(\tilde{X}_1, u)|\} \to 0$ as $\epsilon \to 0$ $\forall\, r, s$.
- A6 $\Lambda = ((\, \mathrm{E}\{\mathrm{D}_s \mathrm{D}_r h(\tilde{X}_1, \mu)\}\,))$ is nonsingular.

Asymptotic Distribution of μ_{F_n} (cont.)

Theorem

Define $\mu_n = \phi(\mu_{F_n})$. *Then under the assumptions* A1-A6,

$$\sqrt{n}(\mu_n - \mu) \xrightarrow{\mathcal{L}} N(0, \Lambda^{-1}\Sigma(\Lambda')^{-1}) \tag{3.1}$$

where $\Sigma = \mathrm{Cov}[\mathrm{D}h(\tilde{X}_1, \mu)]$.

Confidence region for population mean

- An asymptotic confidence set for μ_F of level $(1-\theta)$ given by

$$\{\mu_F : n(\mu_n - \mu)'\hat{\Lambda}'\hat{\Sigma}^{-1}\hat{\Lambda}(\mu_n - \mu) \leq \mathcal{X}_d^2(1-\theta)\}. \qquad (3.2)$$

- The corresponding pivotal bootstrapped confidence region given by

$$\{\mu_F : n(\mu_n - \mu)'\hat{\Lambda}'\hat{\Sigma}^{-1}\hat{\Lambda}(\mu_n - \mu) \leq c^*(1-\theta)\} \qquad (3.3)$$

where $c^*(1-\theta)$ is the upper $(1-\theta)$ quantile of the bootstrapped values of the statistic in (3.2).

Assumptions

- A1 Q has support in a single coordinate patch, (U, ϕ). Let $\tilde{X}_j = \phi(X_j)$, $j = 1, \ldots, n$.
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- A6 $\Lambda = ((\; \mathrm{E}\{\mathrm{D}_s \mathrm{D}_r h(\tilde{X}_1, \mu)\} \;))$ is nonsingular.

Asymptotic Distribution of the Sample Fréchet Variation

Theorem

Under assumptions A1-A6 *and assuming* $\mathrm{E}[\rho^{2\alpha}(X_1, \mu_F)] < \infty$, *one has*

$$\sqrt{n}(V_n - V) \xrightarrow{\mathcal{L}} N(0, \mathrm{var}(\rho^{\alpha}(X_1, \mu_F))) . \qquad (3.4)$$

V_n is a consistent estimator of V even when there there is not a unique population mean, however asymptotic normality breaks down in case of non-uniqueness.

Intrinsic Mean and Variation

- Let (M, g) be a Riemannian manifold with metric tensor g.
- Frechet function of a probability Q is defined as

$$F(p) = \int_M d_g^2(p, m)Q(dm). \tag{4.1}$$

- Fréchet mean (if unique) and variation of Q are called the intrinsic mean and intrinsic variation.

Uniqueness of Intrinsic Mean

- Let M be complete and connected Riemannian manifold of dimension d.
- Define $r_* = \min\{\text{inj}(M), \frac{\pi}{\sqrt{C}}\}$ where $\text{inj}(M)$ is the injectivity radius of M and $C \geq 0$ is an upper bound on sectional curvatures of M.

Proposition (Kendall (1990))

If Q has support in a geodesic ball of radius $\frac{r_*}{2}$, say $B(p, \frac{r_*}{2})$, then it has a unique intrinsic mean μ_I in that ball.

Asymptotic distribution of Sample Intrinsic Mean

- Let μ_{nI} be the sample intrinsic mean from iid sample $X_1, \ldots, X_n \sim Q$.
- Let ϕ be coordinates of $\exp_{\mu_I}^{-1}$: the inverse exponential map into $T_{\mu_I}M$. $\mu_n = \phi(\mu_{nI})$.

Theorem (BB (2008b))

If $\mathrm{supp}(Q) \subseteq B(\mu_I, \frac{r_*}{2})$, *then*

$$\sqrt{n}\mu_n \xrightarrow{\mathcal{L}} N_d(0, \Lambda^{-1}\Sigma\Lambda^{-1})$$

Expressions for Σ, Λ

- Let $\tilde{X}_j = \phi(X_j)$, $j = 1, \ldots, n$.
- If $\mathrm{supp}(Q) \subseteq B(\mu_I, \frac{r_*}{2})$, then Λ is nonsingular and when M has constant curvature C, has the expression

$$\Lambda_{rs} = 2\mathrm{E}\{\left(\frac{1 - f(|\tilde{X}_1|)}{|\tilde{X}_1|^2}\right) \tilde{X}_1^r \tilde{X}_1^s + f(|\tilde{X}_1|)\delta_{rs}\}, \quad 1 \leq r, s \leq d$$

where

$$f(y) = \begin{cases} 1 \text{ if } C = 0 \\ \sqrt{C}y \frac{\cos(\sqrt{C}y)}{\sin(\sqrt{C}y)} \text{ if } C > 0 \\ \sqrt{-C}y \frac{\cosh(\sqrt{-C}y)}{\sinh(\sqrt{-C}y)} \text{ if } C < 0. \end{cases} \tag{4.2}$$

- $\Sigma = 4\mathrm{E}(\tilde{X}_1 \tilde{X}_1')$ is nonsingular when for example Q has a density.

Two Sample Intrinsic Tests: Independent Samples I

- $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ be two iid samples from Q_1 and Q_2 respectively that are mutually independent.
- μ_i and V_i be the intrinsic means and variations of Q_i, $\hat{\mu}_i$ and $\hat{V}_i$ be their sample estimates.
- Want to test $H_0 : \mu_1 = \mu_2 = \mu$ (say) against $H_1 : \mu_1 \neq \mu_2$.
- Under H_0,

$$\sqrt{n}(\phi(\hat{\mu}_1) - \phi(\hat{\mu}_2)) \xrightarrow{\mathcal{L}} N_d\left(0, \Sigma\right), \tag{4.3}$$

ϕ being $\exp_\mu^{-1}$.

Two Sample Intrinsic Tests: Independent Samples II

- Estimate μ by the pooled sample intrinsic mean $\hat{\mu}$.
- $\hat{\phi}$ be $exp_{\hat{\mu}}^{-1}$, μ_{ni} the coordinates of $\hat{\phi}(\hat{\mu}_i)$, $i = 1, 2$.
- Then

$$T_{n1} = n(\mu_{n1} - \mu_{n2})'\hat{\Sigma}^{-1}(\mu_{n1} - \mu_{n2}) \xrightarrow{\mathcal{L}} \mathcal{X}_d^2. \qquad (4.4)$$

- Reject H_0 at asymptotic level α if $T_{n1} > \mathcal{X}_d^2(1-\alpha)$.

Two Sample Intrinsic Tests: Independent Samples III

- $H_0 : V_1 = V_2 = V$ (say) against $H_1 : V_1 \neq V_2$.
- Let $\frac{n_1}{n_1+n_2} \to \theta$, $\sigma_i^2 = \int_M (d_g^2(x, \mu_i) - V_i)^2 Q_i(dx)$.
- Under H_0,

$$\sqrt{n}(\hat{V}_1 - \hat{V}_2) \xrightarrow{\mathcal{L}} N(0, \left(\frac{\sigma_1^2}{\theta} + \frac{\sigma_2^2}{1-\theta}\right)).$$

- Then

$$T_{n2} = \frac{\hat{V}_1 - \hat{V}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \xrightarrow{\mathcal{L}} N(0, 1)$$

  where s_i^2 is the sample analog of σ_i^2.
- Reject H_0 at asymptotic level α if $|T_{n2}| > Z(1 - \frac{\alpha}{2})$.

Two Sample Intrinsic Tests: Matched-Pair Samples

- $(X_1, Y_1), \ldots, (X_n, Y_n)$ is an iid sample from Q on $\bar{M} = M \times M$.
- Q_i, $i = 1, 2$, denote the marginals.
- Let μ_i, V_i denote the means and variations of Q_i, $i = 1, 2$ and $\hat{\mu}_i$, $\hat{V}_i$ denote the sample estimates.

Matched-Pair Samples: Testing Equality of Means

- $H_0 : \mu_1 = \mu_2 = \mu$ (say) against $H_1 : \mu_1 \neq \mu_2$
- Under H_0,

$$\sqrt{n}(\phi(\hat{\mu}_1) - \phi(\hat{\mu}_2)) \xrightarrow{\mathcal{L}} N_d(0, \tilde{\Sigma})$$

where $\phi = \exp_\mu^{-1}$,

$$\tilde{\Sigma} = \Lambda_1^{-1}\Sigma_1\Lambda_1^{-1} + \Lambda_2^{-1}\Sigma_2\Lambda_2^{-1} - (\Lambda_1^{-1}\Sigma_{12}\Lambda_2^{-1} + \Lambda_2^{-1}\Sigma_{21}\Lambda_1^{-1}).$$

- Estimate $\phi(\hat{\mu}_i)$ by μ_{ni} and $\tilde{\Sigma}$ by $\hat{\tilde{\Sigma}}$.
- Then

$$T_{n3} = n(\mu_{n1} - \mu_{n2})'\hat{\tilde{\Sigma}}^{-1}(\mu_{n1} - \mu_{n2}) \xrightarrow{\mathcal{L}} \mathcal{X}_d^2$$

- Reject H_0 at asymptotic level α if $T_{n3} > \mathcal{X}_d^2(1-\alpha)$.

Two Sample Intrinsic Tests: Matched-Pair Samples

- $(X_1, Y_1), \ldots, (X_n, Y_n)$ is an iid sample from Q on $\bar{M} = M \times M$.
- Q_i, $i = 1, 2$, denote the marginals.
- Let μ_i, V_i denote the means and variations of Q_i, $i = 1, 2$ and $\hat{\mu}_i$, $\hat{V}_i$ denote the sample estimates.

Matched-Pair Samples: Testing Equality of Variations

- $H_0 : V_1 = V_2$ against $H_1 : V_1 \neq V_2$.
- Under H_0,

$$\sqrt{n}(\hat{V}_1 - \hat{V}_2) \xrightarrow{\mathcal{L}} N(0, \sigma_1^2 + \sigma_2^2 - 2\sigma_{12})$$

where $\sigma_1^2 = \mathrm{Var}(d_g^2(X_1, \mu_1))$, $\sigma_2^2 = \mathrm{Var}(d_g^2(Y_1, \mu_2))$ and $\sigma_{12} = \mathrm{Cov}(d_g^2(X_1, \mu_1), d_g^2(Y_1, \mu_2))$.

- Then

$$T_{n4} = \frac{\sqrt{n}(\hat{V}_1 - \hat{V}_2)}{\sqrt{s_1^2 + s_2^2 - 2s_{12}}} \xrightarrow{\mathcal{L}} N(0, 1).$$

- Reject H_0 at asymptotic level α if $|T_{n4}| > Z(1 - \frac{\alpha}{2})$.

Extrinsic Analysis

- Let M be a differentiable manifold of dimension d and Q be a probability distribution on M.
- Let $J\colon M \to \mathbb{R}^D$ be an embedding equivariant under some group action H.
- ρ is the distance induced by this embedding. Called the Extrinsic distance.

Definition

$J : M \to \mathbb{R}^D$ is H-equivariant if there exists a group homomorphism $\phi : H \to GL(D, \mathbb{R})$ such that

$$J(hp) = \phi(h)J(p)\ \forall p \in M,\ \forall h \in H.$$

Extrinsic Mean and Variation

Definition

Let Q have finite Fréchet function

$$F(x) = \int_M \rho^2(x, y) Q(dy).$$

The Fréchet mean (if unique) and variation of Q are called its extrinsic mean and extrinsic variation. Given a iid sample $X_1, \ldots, X_n$, the sample extrinsic mean is any measurable selection from the sample extrinsic mean set.

Extrinsic Mean and Variation Computation

- Let $\tilde{\mu}$ be the mean of image $\tilde{Q} = Q \circ J^{-1}$ of Q in $\mathbb{R}^D$.
- Let $\tilde{M} = J(M)$.
- $P_{\tilde{M}}\tilde{\mu}$ = Set of projections of $\tilde{\mu}$ on $\tilde{M}$.

Proposition (BP(2003))

If $P_{\tilde{M}}\tilde{\mu}$ is singleton, then $\mu_E = J^{-1}(P_{\tilde{M}}\tilde{\mu})$ is the extrinsic mean of Q. The extrinsic variation of Q equals

$$V = \int_{E^N} \|x - \tilde{\mu}\|^2 \tilde{Q}(dx) + \|\tilde{\mu} - \mu\|^2$$

Asymptotic Distribution of Sample Extrinsic Mean

- Assume $\tilde{\mu}$ of is a nonfocal point of $\mathbb{R}^D$ (has a unique projection on to $\tilde{M}$).
- Assume that the projection P is a smooth map in a neighborhood of $\tilde{\mu}$.
- Let $\tilde{X}_j = J(X_j)$, $j = 1, 2, \ldots, n$.
- Then

$$\sqrt{n}[P(\bar{\tilde{X}}) - P(\tilde{\mu})] = \sqrt{n}(d_{\tilde{\mu}}P)(\bar{\tilde{X}} - \tilde{\mu}) + o_P(1) \qquad (5.1)$$

Asymptotic Distribution of Sample Extrinsic Mean (cont.)

Proposition (BP(2005))

Denote by T_j, the vector of coordinates of $(d_{\tilde{\mu}}P)(\tilde{X}_j - \tilde{\mu})$ with respect to some orthonormal basis for $T_{P(\tilde{\mu})}\tilde{M}$. Then if $Q \circ J^{-1}$ has finite second moments,

$$\sqrt{n}\bar{T} \xrightarrow{\mathcal{L}} N(0, \Sigma)$$

where $\Sigma = \mathrm{Cov}(T_1)$.

Asymptotic Distribution of Sample Extrinsic Variation

Let V and V_n denote the extrinsic variations of Q and Q_n respectively.

Theorem (Abhishek(2008a))

If Q has extrinsic mean μ_E and if $\mathrm{E}\rho^4(X_1, \mu_E) < \infty$, then

$$\sqrt{n}(V_n - V) \xrightarrow{\mathcal{L}} N(0, \mathrm{Var}(\rho^2(X_1, \mu_E))).$$

Extrinsic Analysis on unit sphere

$$M = S^d = \{p \in \mathbb{R}^{d+1} : p = 1\}.$$

- Embedding J = Inclusion map, Extrinsic distance ρ = Chord distance.
- $\tilde{\mu} = \int_{S^d} xQ(dx) \in \mathbb{R}^{d+1}$.
- Extrinsic mean $\mu_E = \frac{\tilde{\mu}}{\|\tilde{\mu}\|}$ exists iff $\tilde{\mu} \neq 0$.
- Extrinsic variation $V = 2(1 - \|\tilde{\mu}\|)$.
- Sample extrinsic variataion asymptotically Normal iff $\tilde{\mu} \neq 0$.

Two Sample Extrinsic Tests: Independent Samples

- Let $X_1, \dots, X_{n_1}$ and $Y_1, \dots, Y_{n_2}$ be mutually independent iid samples from Q_1 and Q_2.
- Want to test $H_0 : Q_1 = Q_2$ by comparing ex. means and variations.
- Let μ_i be the mean of $\tilde{Q}_i = Q_i \circ J^{-1}$.
- Let $\hat{\mu}_i$ be the sample mean of the embedded samples $\tilde{X}_j$ and $\tilde{Y}_j$.

Independent Samples: Comparing Means

- Under $H_0 : \mu_1 = \mu_2 = \mu$ (say),

$$\sqrt{n}(P(\hat{\mu}_1) - P(\hat{\mu}_2)) = \sqrt{n} d_\mu P(\hat{\mu}_1 - \mu) - \sqrt{n} d_\mu P(\hat{\mu}_2 - \mu) + o_P(1)$$
$$\xrightarrow{\mathcal{L}} N(0, \frac{\Sigma^1}{p_1} + \frac{\Sigma^2}{p_2}). \tag{5.2}$$

  where $n = n_1 + n_2$, $p_i = \lim \frac{n_i}{n}$, Σ^i = covaraiance matrix of coordinates of $d_\mu P(\tilde{X}_j - \mu)$ and $d_\mu P(\tilde{Y}_1 - \mu)$ with respect to some chosen basis for $T_\mu \tilde{M}$.
- Let $\hat{\mu}$ = pooled sample mean estimate μ.
- Denote the coordinates of $\{d_{\hat{\mu}} P(\tilde{X}_j - \hat{\mu})\}$ and $\{d_{\hat{\mu}} P(\tilde{Y}_j - \hat{\mu})\}$ by $\{S_j^1\}$ and $\{S_j^2\}$ and by $\hat{\Sigma}^i$, their sample covariance matrices.

- Then under H_0, the statistic

$$T_1 = (\bar{S}^1 - \bar{S}^2)' \left(\frac{1}{n_1} \hat{\Sigma}^1 + \frac{1}{n_2} \hat{\Sigma}^2 \right)^{-1} (\bar{S}^1 - \bar{S}^2) \xrightarrow{\mathcal{L}} \mathcal{X}_d^2. \tag{5.3}$$

- Hence reject H_0 at asymptotic level α if $T_1 > \mathcal{X}_d^2(1 - \alpha)$.

Comparing Extrinsic Means

- $H_0 : \mu_{1E} = \mu_{2E}$ against $H_a : \mu_{1E} \neq \mu_{2E}$.
- Let L be the linear projection from $\mathbb{R}^D$ on to $T_{\hat{\mu}}\tilde{M}$, L_i are the linear projection on to $T_{\hat{\mu}}\tilde{M}$ from $T_{P(\hat{\mu}_i)}\tilde{M}$, $\hat{\Sigma}_i$ the sample covariance matrices of $\{d_{\hat{\mu}_1}P(\tilde{X}_j - \hat{\mu}_1)\}_{j=1}^{n_1}$ and $\{d_{\hat{\mu}_2}P(\tilde{Y}_j - \hat{\mu}_2)\}_{j=1}^{n_2}$ and

$$\hat{\Sigma} = \frac{1}{n_1}L_1\hat{\Sigma}_1L_1' + \frac{1}{n_2}L_2\hat{\Sigma}_2L_2'$$

- Under H_0,

$$T_2 = L[P(\hat{\mu}_1) - P(\hat{\mu}_2)]'\hat{\Sigma}^{-1}L[P(\hat{\mu}_1) - P(\hat{\mu}_2)] \xrightarrow{\mathcal{L}} \mathcal{X}_d^2 \quad (5.4)$$

- Reject H_0 at asymptotic level α if $T_2 > \mathcal{X}_d^2(1-\alpha)$.

Independent Samples: Comparing Extrinsic Variations

- $H_0 : V_1 = V_2$ against $H_a : V_1 \neq V_2$.
- Under H_0,

$$T_3 = \frac{\hat{V}_1 - \hat{V}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \xrightarrow{\mathcal{L}} N(0,1) \tag{5.5}$$

  where s_i^2 are sample variances of the ex. dist. squared fromthe repective ex. means.
- Reject H_0 if $|T_3| > Z\left(1 - \frac{\alpha}{2}\right)$.

Kendall's (Direct) Similarity Shape Space Σ_m^k [Kendall(1984)]

- Preshape of a k-ad $x \in \mathbb{R}^{m \times k}$ is

$$z = (x_1 - \bar{x}, x_2 - \bar{x}, \dots, x_k - \bar{x}) / \|x - \bar{x}\|$$

  where

$$\bar{x} = \frac{1}{k} \sum_{j=1}^{k} x_j, \ \|A\| = \sum_{i=1}^{m} \sum_{j=1}^{k} a_{ij}^2 = \mathrm{Trace}(AA').$$

- Set of all preshapes is the Preshape Sphere $S_m^k \equiv S^{km-m-1}$.

Similarity Shape Space Σ_m^k

- Similarity shape of x is

$$\sigma(x) = \pi(z) = \{Az \colon A \in SO(m)\}.$$

- Σ_m^k is the set of all $\pi(z)$ where $\text{rank}(z) \geq m - 1$,

$$\Sigma_m^k = NS_m^k / SO(m).$$

- $\pi : NS_m^k \to \Sigma_m^k$ is a Riemannian submersion.
- Hence Σ_m^k is a Riemannian manifold of dimension $km - m - \frac{m(m-1)}{2}$.

Geometry of Σ_2^k

- k-ad x represented by a complex k-vector ($\in \mathbb{C}^k$).
- Let z be its preshape. ($z \in \mathbb{C}^k$, $\|z\| = 1$, $z'\mathbf{1}_k = 0$)
- Shape of x is

$$\sigma(x) = \pi(z) = \{e^{i\theta} z : \ -\pi < \theta \leq \pi\}.$$

- $\Sigma_2^k \simeq S^{2k-3}/S^1 \simeq \mathbb{C}P^{k-2}$.

Intrinsic Mean

- Identified with $\mathbb{C}P^{k-2}$, Σ_2^k is compact Riemannian manifold.
- All sectional curvatures between 1 and 4.
- Injectivity radius = $\frac{\pi}{2}$.
- If $\mathrm{supp}(Q) \subseteq B(p, \frac{\pi}{4})$, Q has a unique intrinsic mean in that ball.

Asymptotic Distribution of Sample Intrinsic Mean

- Assume $\mathrm{supp}(Q) \subseteq B(p, \frac{\pi}{4})$ and let μ_I be the unique intrinsic mean of Q in that ball.
- Let μ_n be the coordinates of the sample intrinsic mean under $\exp_{\mu_I}^{-1}$.

Theorem (BB(2008b))

If $\mathrm{supp}(Q) \subseteq B(\mu_I, R)$, *$R$ being the unique solution of* $\tan(x) = 2x$, $x \in (0, \frac{\pi}{2})$, *then* $\sqrt{n}\mu_n \xrightarrow{\mathcal{L}} N(0, \Lambda^{-1}\Sigma\Lambda^{-1})$ *and* Λ *is positive definite.*

$R \in (\frac{\pi}{3}, \frac{2\pi}{5})$, is approximately 0.37101π.

Extrinsic Analysis on Σ_2^k

- Σ_2^k can be embedded into $S(k, \mathbb{C})$: space of $k \times k$ Hermitian matrices, by the Veronese-Whitney embedding

$$J : \Sigma_2^k \to S(k, \mathbb{C}), \; J(\sigma(x)) = J(\pi(z)) = zz^*.$$

- If $S^+(k, \mathbb{C})$ = complex p.s.d matrices, then

$$J(\Sigma_2^k) = \{A \in S^+(k, \mathbb{C}) : \mathrm{rank}(A) = 1, \; \mathrm{Trace}(A) = 1, A\mathbf{1}_k = 0\}.$$

- J is H-equivariant, $H = SU(k)$: $k \times k$ matrices A, $AA^* = I_k$, $\det(A) = +1$.

Extrinsic Mean and Variation on Σ_2^k

Let $\tilde{\mu} =$ Mean of $Q \circ J^{-1}$: p.s.d of complex rank ≥ 1, trace = 1, $\tilde{\mu}\mathbf{1}_k = 0$.

Proposition (BP(2003))

The ex. mean μ_E exists iff the largest eigenvalue λ_k of $\tilde{\mu}$ is simple. Then $\mu_E = \pi(\mu)$, μ being a unit eigenvector corresponding to λ_k. The ex. variation $V = 2(1 - \lambda_k)$.

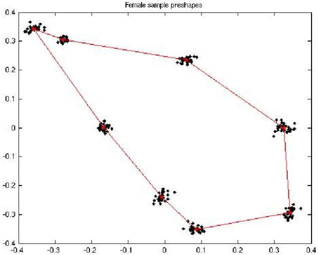
Asymptotic Distribution of Sample Extrinsic Mean and Variation

- Assume μ_E exists.
- Then the sample ex. mean μ_{nE} is a consistent estimator of μ_E.
- The projection map $P : \tilde{\mu} \mapsto \mu\mu^*$ is smooth near $\tilde{\mu}$ and from Proposition 5.2, it follows that $J(\mu_{nE})$ has an asymptotic Normal distribution.
- Fom Theorem 5.3, it follows that the sample ex. variation has an asymptotic Normal distribution.

Gorilla Skulls

- 8 landmarks chosen on the midline plane of the 2D image of 29 male and 30 female gorilla skulls (Dryden and Mardia (1998)).
- Goal: Study the shapes of the skulls and use that to detect difference in shapes between the sexes. Finds application in morphometrics.
- Two mutually independent iid sample of planar shapes of sizes 29 and 30 on Σ_2^k, $k = 8$.

Gorilla Skull Images: Sample k-ads



8 LANDMARKS FROM SKULLS OF 30 FEMALE & 29 MALE GORILLAS ALONG WITH THE EXTRINSIC MEANS

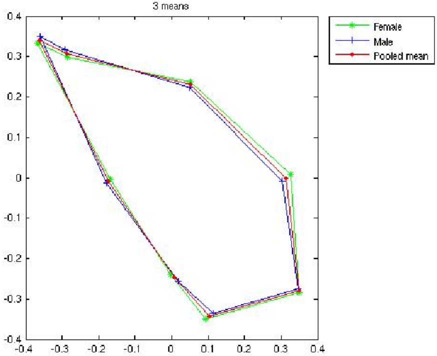
Gorilla Skulls: Comparing Extrinsic Means

- Extrinsic test statistics defined through equations (5.3) & (5.4) for comparing the extrinsic mean shapes and corresponding asymptotic p-values are

$$T_1 = 392.6, \text{ p-value } = P(\mathcal{X}_{12}^2 > 392.6) < 10^{-16},$$
$$T_2 = 392.06, \text{ p-value } < 10^{-16}.$$

- Also compared the mean shapes by pivotal bootstrap method using the b.s. analogue of T_2. P-value for the bootstrap test using 10^5 simulations turns out to be 0.
- Reject H_0 that the two sexes have the same extrinsic mean shape.

Gorilla Skulls: Extrinsic Mean Shapes Plot



SAMPLE EX. MEANS FOR THE TWO SEXES ALONG WITH THE POOLED SAMPLE EX. MEAN

Gorilla Skulls: Comparing Extrinsic Variations

- Sample extrinsic variation for female sample = 0.0038, for male = 0.005.
- Test statistic for testing equality of variations from (5.5) is 0.923 and the asymptotic p-value using standard Normal approximation is 0.356.
- Accept H_0 that the two sexes have the same extrinsic variation.

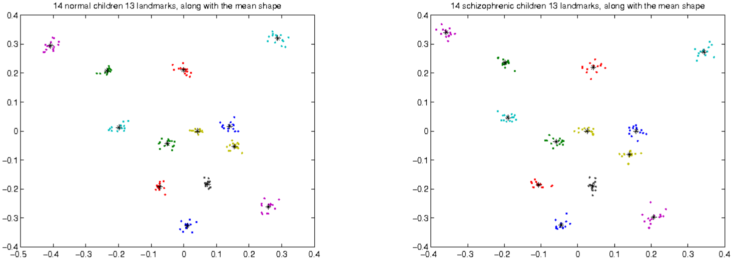
Gorilla Skulls: Intrinsic Analysis

- Sample intrinsic means very close to their extrinsic counterparts, the geodesic distance between the intrinsic and extrinsic means being 5.54×10^{-7} for the female sample and 1.96×10^{-6} for the male sample.
- Intrinsic test statistic from (4.4) equals $T_{n1} = 391.63$.
- Asymptotic p-value = $P(\mathcal{X}_{12}^2 > 391.63) < 10^{-16}$.
- Reject H_0 that the two sexes have the same intrinsic mean shape.

Schizophrenia Detection

- 13 landmarks for 14 schizophrenic & 14 normal children on a midsagittal 2D slice from MR brain scan. (Bookstein(1991))
- 2 independent samples of size 14 on Σ_2^k, $k = 13$ (dimension = 22).
- Goal: Study differences in shapes of brains between the two groups which can be used to detect schizophrenia.

Schizophrenia plots: Sample k-ads



13 LANDMARKS FROM BRAIN SCAN OF 14 NORMAL & 14 SCHIZOPHRENIC CHILDREN ALONG WITH THE EXTRINSIC MEANS

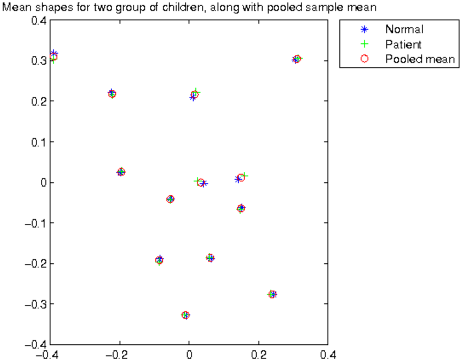
Schizophrenia Detection: Extrinsic Analysis

- Extrinsic test statistics defined through equations (5.3) & (5.4) for comparing the extrinsic mean shapes and corresponding asymptotic p-values are

$$T_1 = 95.5476, \text{ p-value } = P(\mathcal{X}_{22}^2 > 95.5476) = 3.8 \times 10^{-11},$$
$$T_2 = 95.2549, \text{ p-value } = 4.3048 \times 10^{-11}.$$

- Extrinsic sample variations for patient and normal samples are 0.0107 and 0.0093 respectively.
- Test statistic for testing equality of extrinsic variations from (5.5) is 0.9461 and the asymptotic p-value using standard Normal approximation is 0.3441.

Schizophrenia plots: Extrinsic mean shapes



SAMPLE EX. MEANS FOR TWO GROUP OF CHILDREN ALONG WITH THE POOLED SAMPLE EX. MEAN

Schizophrenia Detection: Extrinsic Analysis by Bootstrap Methods

- Since the dimension 22 of the underlying shape space is much higher than the sample sizes (14), the bootstrap estimate of the standard error tends to be singular in most simulations.
- Compare the first five principal scores of the coordinates of the sample extrinsic means which explain 72% of total variation.
- The bootstrap p-value from 10^4 simulations equals 0.0168.
- Asymptotic p-value using $\mathcal{X}_5^2$ approximation = 0.0323.

Bootstrap Method 2

- Alternatively estimate the standard error from bootstrap sample means and perform a asymptotic chi-squared test.
- Value of T_2 then = 105.955.
- Asymptotic p-value = $P(\mathcal{X}_{22}^2 > 105.955) = 5.8 \times 10^{-13}$.
- Bootstrap p-value for comparing variations with 10^4 simulations equals 0.3564.

Schizophrenia Detection: Intrinsic Analysis

- Sample intrinsic means very close to their extrinsic counterparts, the geodesic distance between the intrinsic and extrinsic means being 1.65×10^{-5} for the normal children sample and 4.29×10^{-5} for the sample of schizophrenic children.
- Intrinsic test statistic from (4.4) equals $T_{n1} = 95.4587$.
- Asymptotic p-value = $P(\mathcal{X}_{22}^2 > 95.4587) = 3.97 \times 10^{-11}$.

Schizophrenia Detection: Conclusions

- Reject H_0 that the two group of children have the same mean shape (both extrinsic and intrinsic) at asymptotic levels greater than or equal to 10^{-10}.
- Accept H_0 that the two groups have the same ex. variation in shapes at levels of significance less than or equal to 0.3.
- Bootstrap methods yield consistent results.
- Since the mean shapes are different, conclude that the probability distributions of the shapes of brain scans of normal and schizophrenic children are distinct.

Geometry of $R\Sigma_m^k$

- Recall the similarity shape of a centered norm 1 k-ad z and the Similarity Shape Space Σ_m^k as in slide 41.
- $R\Sigma_m^k = S_m^k/O(m)$.
- For $O(m)$ to act freely, assume $\text{rank}(z) = m$ and redefine $R\Sigma_m^k$ as $NS_m^k/O(m)$.
- Then $R\Sigma_m^k$ is a Riemannian manifold (not complete).

Extrinsic Analysis on $R\Sigma_m^k$

- Can be embedded into $S(k, \mathbb{R})$ via

$$J : R\Sigma_m^k \to S(k, \mathbb{R}),\ J(\sigma(z)) = z'z.$$

- J is equivariant under the action of $O(k)$.
- $J(R\Sigma_m^k) = \{A\colon A = A'$, A is p.s.d., Rank(A) = m, Trace(A) = 1, $A\mathbf{1}_k = 0\}$.
- $J(R\Sigma_m^k)$ submanifold of $S(k, \mathbb{R})$ (not complete).

Extrinsic Mean on $R\Sigma_m^k$

$\tilde{\mu} =$ Mean of $Q \circ J^{-1}$ in $S(k, \mathbb{R})$: p.s.d. of rank $\geq m$, Trace = 1, $\tilde{\mu}\mathbf{1}_k = 0$.

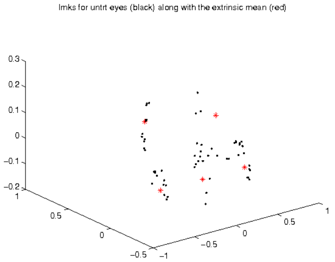
Theorem (Abhishek(2008))

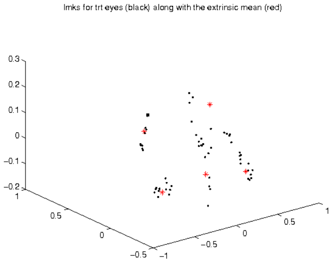
Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ be eigenvalues of $\tilde{\mu}$ and $V_1, V_2, \ldots, V_k$ be corresponding orthonormal eigenvectors. Let $\bar{\lambda} = \frac{\sum_{j=1}^m \lambda_j}{m}$.
(i) μ_E exists iff $\lambda_m > \lambda_{m+1}$ and then
(ii) $J(\mu_E) = \sum_{j=1}^m (\lambda_j - \bar{\lambda} + \frac{1}{m}) V_j V_j'$, $\mu_E = \sigma(z)$ where $z = [z_1, \ldots, z_m]'$, $z_j = \sqrt{\lambda_j - \bar{\lambda} + \frac{1}{m}} V_j$.

Application: Glaucoma detection

- 5 landmarks on the glaucoma induced eye and the normal eye of 12 rhesus monkeys. (BP(2005))
- Paired sample of size 12 on $R\Sigma_3^k$, $k = 5$ (Dimension = 8).
- Goal: Test for any significant difference between the shapes of the glaucoma induced and normal eyes by comparing the ex. means and variations.

Glucoma detection plots: sample *k*-ads



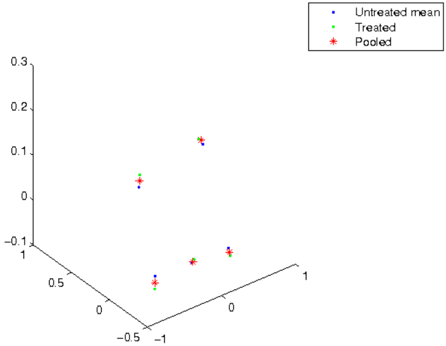


5 LANDMARKS FROM THE NORMAL & INFECTED EYES OF 12 MONKEYS ALONG WITH THE SAMPLE EXTRINSIC MEANS

Glaucoma detection: Comparing Extrinsic Means

- Test statistic for comparing the extrinsic mean shapes = 36.56. (Abhishek(2008))
- Asymptotic p-value = $P(\mathcal{X}_8^2 > 36.56) = 1.38 \times 10^{-5}$.
- 1st two P.C.'s explain more than 80% of variation in shape.
- Bootstrap p-value for comparing the 1st two P.C.S of the mean shapes from 10^4 simulations = 0.0098.
- Asymptotic p-value using $\mathcal{X}_2^2$ approximation = 0.002.

Glaucoma Detection Plots: Sample Extrinsic Means



EX. MEAN SHAPES FOR THE 2 EYES ALONG WITH POOLED SAMPLE EX. MEAN

Glaucoma Detection: Extrinsic Variation Comparision

- Sample extrinsic variation for glaucoma induced eye = 0.038.
- Sample extrinsic variation for normal eye = 0.041.
- Test statistic for testing equality of extrinsic variations = -0.5572. (Abhishek(2008))
- Asymptotic p-value = $P(|Z| > 0.5572) = 0.577$.
- Bootstrap p-value from 10^4 simulations = 0.59.

Glaucoma Detection: Conclusions

- Conclude at asymptotic level 0.0001 or higher that the mean shapes of the two eyes are significantly different.
- Also conclude at levels 0.01 or higher that the 1st two principal coordinates of the mean shapes are different, both by asymptotic and pivotal bootstrap methods.
- Conclude that the ex. variations are equal at levels 0.5 or lower both by asymptotic and pivotal bootstrap methods.
- Since the mean shapes for the two eyes are found to be different, conclude that the underlying probability distributions are distinct and hence Glaucoma indeed changes the shape of the eyes.

Geometry of $A\Sigma_m^k$

- The affine shape of a k-ad x with landmarks in $\mathbb{R}^m$ is its orbit under the group of all affine transformations.
- $A\Sigma_m^k = H(m,k)/GL(m,\mathbb{R})$ where $H(m,k)$ consists of all centered k-ads.
- Affine shape of a centered k-ad $u \in H(m,k)$,

$$\sigma(u) = \{Au : \; A \in GL(m,\mathbb{R})\}$$

is the subspace spanned by the m rows of u.

Geometry of $A\Sigma_m^k$ contd...

- For the action of $GL(m, \mathbb{R})$ to be free and $A\Sigma_m^k$ to be a manifold, redefine $A\Sigma_m^k$ to be the affine shapes of those centered k-ads u whose lmks. span $\mathbb{R}^m$ and hence rank(u) = m.
- Then $A\Sigma_m^k$ can be identified with the Grassmanian $G_m(k-1)$: the set off all m dimensional subspaces of $\mathbb{R}^{k-1}$ (Sparr(1992)).
- Manifold of dimension $km - m - m^2$.

Extrinsic Analysis on Affine Shape Spaces

- $A\Sigma_m^k$ can be embedded into $S(k, \mathbb{R})$ via $J : A\Sigma_m^k \to S(k, \mathbb{R})$, J maps the m dimensional subspace associated with the affine shape of a centered k-ad u to the unique linear projection matrix on to that subspace.
- $J(\sigma(u)) = u'(uu')^{-1}u$.
- J is equivariant under the action of $O(k)$ (Dimitric(1996)).
- $J(A\Sigma_m^k) = \{A \in S(k, \mathbb{R}) \colon A^2 = A, \ \text{trace}(A) = m, \ A\mathbf{1}_k = 0\}$ is a compact Riemannian submanifold of $S(k, \mathbb{R})$.

Extrinsic Mean and Variation

$\tilde{\mu} =$ Mean of $Q \circ J^{-1}$ in $S(k, \mathbb{R})$: p.s.d. of rank $\geq m$.

Proposition (Sughatadasa(2006))

Let $U_1, \ldots, U_k$ be orthonormal eigenvectors of $\tilde{\mu}$ corresponding to ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k$.
(i) μ_E exists iff $\lambda_m > \lambda_{m+1}$ and then
(ii) $J(\mu_E) = \sum_{j=1}^m U_j U_j'$, $\mu_E = \sigma(F')$ where $F = (U_1 \ldots U_m)$.

Ex. variation $V = 2(m - \sum_{j=1}^m \lambda_j)$.

Asymptotic Distribution of Sample Extrinsic Mean

- $X_1, \ldots, X_n$ iid sample from Q, $\tilde{X}_j = J(X_J)$.
- μ_{nE}: sample ex. mean.

$$\sqrt{n}[J(\mu_{nE}) - J(\mu_E)] = \sqrt{n} d_{\tilde{\mu}} P(\bar{\tilde{X}} - \tilde{\mu}) + o_P(1) \xrightarrow{\mathcal{L}} N(0, \Sigma)$$

Asymptotic Distribution of μ_{nE} (contd.)

$\Sigma = \text{Cov}(T_1)$, T_j being coordinates of $d_{\tilde{\mu}}P(\tilde{X}_j - \tilde{\mu})$ which are

$$T_j = ((T_j)_{ab} : \ 1 \leq a \leq m < b < k)$$
$$(T_j)_{ab} = \sqrt{2}(\lambda_a - \lambda_b)^{-1} U_a' \tilde{X}_j U_b.$$

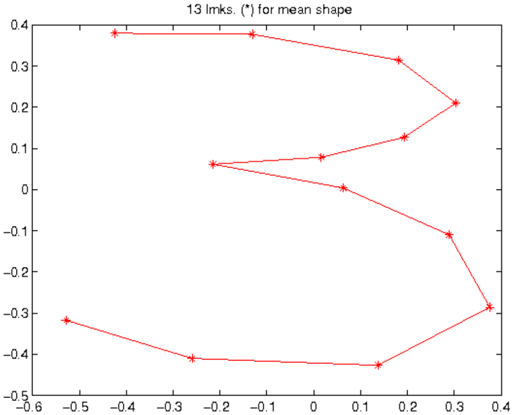
Application to Handwritten Digit Recognition

- Random sample of 30 handwritten digit '3' collected and 13 landmarks recorded on each image by Anderson(1997).
- Objective: Analyse the affine shape of the sample points and estimate the mean shape and variation.
- Application: This can be used as a prior model for digit recognition from images of handwritten codes.
- Sample of size 30 on $A\Sigma_m^k$, $k = 13$, $m = 2$.

Digit ‘3’ Recognition: Extrinsic Mean and Variation

- Sample ex. variation of 0.27 fairly large.
- Remove 2 outliers and recompute the sample ex. mean and variation.
- The sample variation now turns out to be 0.19.

Sample Extrinsic Mean of Digit '3' Plot



Extrinsic mean shape for handwritten digit 3 sample

Confidence Region for True Mean Shape

- Dim. of $A\Sigma_2^{13}$ is 20. An asymptotic 95% C.R. for the ex. mean μ_E from $\mathcal{X}_{20}^2$ approximation given by

$$\{\mu_E : nL[P(\tilde{\mu}) - P(\bar{\tilde{X}})]'\hat{\Sigma}^{-1}L[P(\tilde{\mu}) - P(\bar{\tilde{X}})] \leq 31.4104\}.$$

  L denotes linear projection into tangent space of $J(A\Sigma_2^{13})$ at $P(\bar{\tilde{X}})$.
- Bootstrap C.R. from 10^5 simulations has cut-off point of 1.077.

Confidence Interval for True Variation in Shape

- A 95% C.I. for the ex. variation V by Normal approximation is $V \in [0.140, 0.243]$.
- Pivotal bootstrap C.I. using 10^5 simulations turns out to be $[0.119, 0.264]$.

Real Projective Space $\mathbb{R}P^m$

- is the set of all lines through origin in $\mathbb{R}^{m+1}$.
- Can be identified with S^m/G, G comprising the identity map and the antipodal map $p \mapsto -p$.
- Its structure as a m-dimensional manifold (with quotient topology) and its Riemannian structure both derive from this identification.

Intrinsic Analysis on $\mathbb{R}P^m$

- Constant sectional curvature of 4.
- If $\mathrm{supp}(Q) \subseteq B(p, \frac{\pi}{4})$, unique intrinsic mean μ_I.
- Let $\mu_n = \phi(\mu_{nI})$ be the coordinates of sample intrinsic mean under $exp_{\mu_I}^{-1}$.
- if $\mathrm{supp}(Q) \subseteq B(\mu_I, \frac{\pi}{4})$, from slides 20 & 21, it follows that

$$\sqrt{n}\mu_n \xrightarrow{\mathcal{L}} N(0, \Lambda^{-1}\Sigma\Lambda^{-1})$$

Extrinsic Analysis on $\mathbb{R}P^m$

- Can be embedded into $S(m+1,\mathbb{R})$ via the Veronese-Whitney embedding J, $J([u]) = uu'$, $u \in S^m$. Introduced by Watson (1983).
- Denotes the the space of all $(m+1)\times(m+1)$ p.s.d matrices as $S^+(m+1,\mathbb{R})$. Then

$$J(\mathbb{R}P^m) = \{A \in S^+(m+1,\mathbb{R}) : \mathrm{rank}(A) = \mathrm{Trace}(A) = 1\}$$

  compact Riemannian submanifold of $S(m+1,\mathbb{R})$ of dimension m.
- J equivariant under the action of $O(m+1)$.

Extrinsic Mean and Variation

$\tilde{\mu}$ = Mean of $Q \circ J^{-1}$ in $S(m+1, \mathbb{R})$: p.s.d. of rank ≥ 1.

Proposition (BP(2003))

Let λ_{m+1} denote the largest eigenvalue of $\tilde{\mu}$ and μ be a corresponding unit eigenvector.
(i) μ_E exists iff λ_{m+1} is a simple eigenvalue and then
(ii) $\mu_E = [\mu]$.

Ex. variation $V = 2(1 - \lambda_{m+1})$.

Asymptotic Distribution of the Sample Extrinsic Mean

- Assume $\tilde{\mu}$, the mean of $Q \circ J^{-1}$ is nonfocal, i.e. has simple largest eigenvalue, so that ex. mean μ_E of Q exists.
- Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m < \lambda_{m+1}$ be the eigenvalues of $\tilde{\mu}$ and $U_1, U_2, \ldots, U_m, U_{m+1}$ be corresponding orthonormal eigenvectors.
- Let μ_{nE} be the sample ex. mean from a iid random sample $X_1, \ldots, X_n$ from Q on $\mathbb{R}P^m$.
- Let $\tilde{X}_j = J(X_j)$ and $\bar{\tilde{X}} = \frac{1}{n}\sum_{j=1}^{n} \tilde{X}_j$.

Asymptotic Distribution of μ_{nE} (contd..)

- $P : \tilde{\mu} \mapsto U_{m+1}U'_{m+1}$ is smooth in aneighborhood of $\tilde{\mu}$.
- Let T_j be the coordinates of $d_{\tilde{\mu}}P(\tilde{X}_j - \tilde{\mu})$, $j = 1, \ldots, n$.
- Then $T_j = (T_j^1, \ldots, T_j^m)$ where

$$T_j^a = \sqrt{2}(\lambda_{m+1} - \lambda_a)^{-1} U'_a \tilde{X}_j U_{m+1}, \;\; a = 1, \ldots, m.$$

and

$$\sqrt{n}[J(\mu_{nE}) - J(\mu_E)] = \sqrt{n}\bar{T} + o_P(1) \xrightarrow{\mathcal{L}} N(0, \Sigma)$$

where $\Sigma = \mathrm{Cov}(T_1)$.

Projective Shape Space $P\Sigma_m^k$

- Landmarks of the k-ad viewed in $\mathbb{R}P^m$.
- Projective transformation α on $\mathbb{R}P^m$ defined by $A \in GL(m+1, \mathbb{R})$:
$$\alpha([y]) = [Ay]$$
where $y = (y^1, \ldots, y^{m+1})' \in \mathbb{R}^{m+1} \setminus \{0\}$, $[y]$ is the line through y:
$$[y] = \{\lambda y \colon \lambda \neq 0\}.$$
- $PGL(m)$ is the group of all projective transformations on $\mathbb{R}P^m$.

$P\Sigma_m^k$

- For a k-ad $x = (x_1, \ldots, x_k)$ in $(\mathbb{R}P^m)^k$, its projective shape is its orbit under $PGL(m)$, i.e.,

$$\sigma(x) = \alpha(x) \doteq \{(\alpha x_1, \ldots, \alpha x_k) \colon \alpha \in PGL(m)\}.$$

- A k-ad x is in general position if it has atleast $m+2$ landmarks, the linear span of any $m+1$ points from which is $\mathbb{R}P^m$.
- The space of projective shapes of all k-ads in general position is $P\Sigma_m^k$.
- Riemannian manifold of dimension $m(k-m-2)$.

Projective Shape Space: Identification with $(\mathbb{R}P^m)^{k-m-2}$

- A projective frame is an ordered system of $m+2$ points in $\mathbb{R}P^m$ in general position.
- Standard frame defined as

$$e = ([e_1], \ldots, [e_{m+1}], [e_1 + e_2 + \ldots + e_{m+1}]),$$

  $e_j \in \mathbb{R}^{m+1}$ has 1 in the j-th coordinate and 0 elsewhere.
- Given any projective frame y, there exists a unique $\alpha \in PGL(m)$ such that $\alpha(y) = e$.

Identification with $(\mathbb{R}P^m)^{k-m-2}$ (contd..)

- By ordering the points in a k-ad such that the first $m+2$ points are in general position, one may bring this ordered set to the standard form by a unique $\alpha \in PGL(m)$.
- Then the ordered set of remaining $k-m-2$ points is transformed to a point in $(\mathbb{R}P^m)^{k-m-2}$.
- This provides a diffeomorphism between a open dense subset of $P\Sigma_m^k$ and $(\mathbb{R}P^m)^{k-m-2}$ (Mardia and Patrangenaru (2005)).
- By developing corresponding inference tools on $\mathbb{R}P^m$, one can perform statistical inference in this subset of $P\Sigma_m^k$.

THANK YOU!