INTRINSIC MEAN ON MANIFOLDS

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Overview

- Properties of Intrinsic mean on Riemannian manifolds have been presented.
- The results have been applied to certain manifolds including the planer shape space.

Frechet Mean on Metric Spaces

- $(\mathbf{M}, \rho)$ a metric space and $\mathbf{Q}$ a probability measure on M. The **Frechet function** of Q is

$$F(p) = \int_M \rho^2(p, x) Q(dx), \quad p \in M.$$

- The **Frechet Mean** set of Q is the set of all p for which F(p) is the minimum.

- Suppose every closed and bounded subset of M is compact. If the Frechet function is finite for some p, then the Frechet mean set is nonempty and compact.

Sample Frechet Mean

- $X_1, X_2, \ldots, X_n$ are iid with commom distribution Q, and $Q_n \doteq \frac{1}{n}\sum_{j=1}^{n}\delta_{X_j}$ is the corresponding empirical distribution.

- The Frechet mean set of Q_n is called the **sample (Frechet)mean** set.

- If the Frechet mean of Q exists (as a unique minimizer of F), then every measurable selection from the Frechet sample mean set is a strongly consistent estimator of the Frechet mean of Q.

Intrinsic Means

- Let $(\mathbf{M}, \mathbf{g})$ be a d-dimensional connected complete Riemannian manifold, g being the riemannian metric. Let $\rho = d_g$, the geodesic distance under g.

- Let Q be a probability distribution on M. The Frechet mean set of Q is called its **intrinsic mean** set.

- $X_1, X_2, \ldots, X_n$ iid with commom distribution Q. The sample Frechet mean (set) is called the **sample intrinsic mean** (set).

Existence (Uniqueness) of Intrinsic means

Suppose all sectional curvatures on M bounded above by $\mathbf{C} \geq 0$. Q is a probability distribution on M with finite Frechet function. Support of Q contained in a ball of radius r (wrt d_g) where

$$r = \begin{cases} \infty & \text{if } C = 0 \\ \frac{\pi}{4\sqrt{C}} & \text{if } C > 0 \end{cases}$$

Then the Frechet function, F of Q is strictly convex and hence the intrinsic mean exists (as a unique minimizer of F).

Asymptotic distribution of sample mean

- Assume support of Q contained in a closed geodesic ball $\overline{B}_r(p)$ with center p and radius r, r as above; which is disjoint from the cutlocus $C(p)$.

- Let $\phi = Exp_p^{-1} : B_r(p) \longrightarrow T_pM (\approx \mathbb{R}^d)$.

- Define $h(x,y) = d_g^2(\phi^{-1}x, \phi^{-1}y);\ \ x, y \in \mathbb{R}^d$. Let $((D_r h))_{r=1}^d$ and $((D_r D_s h))_{r,s=1}^d$ be the matrix of first and second order derivatives of $y \mapsto h(x,y)$.

- Let $Y_j = \phi(X_j); j = 1, \ldots, n$; $X_1, \ldots, X_n$ being iid observations from Q. $\mu = \phi(\mu_I)$, μ_I being the intrinsic mean of Q. $\mu_n = \phi(\mu_{n,I})$, $\mu_{n,I}$ being a measurable selection from the sample mean set.

- $\Lambda = E((D_r D_s h(Y_1, \mu)))_{r,s=1}^{d}$;
  $\Sigma = Cov((D_r h(Y_1, \mu)))_{r=1}^{d}$.
  Then Λ and Σ are positive definite and

- $\sqrt{n}(\mu_n - \mu) \xrightarrow{\mathcal{L}} N(0, \Lambda^{-1}\Sigma\Lambda^{-1})$

Expressions for Λ and Σ

When M has constant sectional curvature C, under the above assumptions with $p = \mu_I$, we get:

$$\begin{aligned}
&D_r h(Y_1, \mu) = -2Y_1^r \\
&E(D_r h(Y_1, \mu)) = 0 \\
&\Sigma = 4Cov(Y_1^r, Y_1^s) \\
&\Lambda = 2E((\left(\frac{1 - f|Y_1|}{|Y_1|^2}\right) Y_1^r Y_1^s + (f|Y_1|)\delta_{rs})), \\
&|Y_1| = \sqrt{(Y_1^1)^2 + (Y_1^2)^2 + \ldots (Y_1^d)^2}
\end{aligned}$$

where

$$f(x) = \begin{cases} 1 & \text{if } C = 0 \\ \sqrt{C}x\frac{cos(\sqrt{C}x)}{sin(\sqrt{C}x)} & \text{if } C > 0 \\ \sqrt{-C}x\frac{cosh(\sqrt{-C}x)}{sinh(\sqrt{-C}x)} & \text{if } C < 0 \end{cases}$$

Examples

Example 1(S^{k-1})

- Consider the unit sphere in $\mathbb{R}^k$, namely, $S^{k-1} = \{p \in \mathbb{R}^k : \|p\| = 1\}$.

- The geodesic distance with respect to the scaler product on $\mathbb{R}^k$ is

$$d_g(p, q) = \cos^{-1}(p.q) \in [0, \pi]$$

- It has constant sectional curvature 1. Hence the intrinsic mean of a probability distribution Q exists if its support is contained in a ball of radius atmost $\frac{\pi}{4}$.

- In that case the sample Frechet mean based on a random sample is consistent.

Example 2(Real Projective space $\mathbb{R}P^{k-1}$)

- All lines through the origin in $\mathbb{R}^k$.

- If $x(\neq 0) \in \mathbb{R}^k$, the line through it represented by $[x] = \{\frac{x}{\|x\|}, -\frac{x}{\|x\|}\}$.

- Embedded into $\mathbb{R}^{k^2}$ via the **Veronese-Whitney embedding** $\phi([u]) = uu^{'}$. This induces a Riemannian structure on this space, known as the **Fubini-Study** metric.

- The geodesic distance for this metric is

$$d_g([x],[y]) = \text{arccos}(|u'v|) \in [0, \frac{\pi}{2}]$$

$$u = \frac{x}{\|x\|},\ v = \frac{y}{\|y\|}.$$

- Constant sectional curvature 4.

- Hence the intrinsic mean of a probability distribution Q exists if its support contained in ball of radius atmost $\frac{\pi}{8}$.

Example 3(Planer shape space Σ_2^k)

- Consider k points on the plane not all same. Assume $k > 2$ and refer to such a set as a k-ad.
- A k-ad can be represented by a complex k-vector.
- Shape of a complex k-ad $z = (z_1, \dots z_k)$ is the orbit of z under translation, scaling and rotation. Can be represented by $[\mathbf{z}] = \{e^{i\theta}\frac{(\mathbf{z}-\langle\mathbf{z}\rangle)}{\|\mathbf{z}-\langle\mathbf{z}\rangle\|}: 0 \leq \theta < 2\pi\}$.

- Embedded into $\mathbb{C}^{k^2}$ via the **Veronese-Whitney embedding** $\phi([u]) = uu^*$. This induces a Riemannian structure on this space, known as the **Fubini-Study** metric.

- The geodesic distance for the Fubini-Study metric is

$$d_g([z],[w]) = \arccos(|u'\overline{v}|) \in [0, \frac{\pi}{2}]$$
$$u = \frac{(\mathbf{z} - \langle \mathbf{z} \rangle)}{\|\mathbf{z} - \langle \mathbf{z} \rangle\|},\ v = \frac{(\mathbf{w} - \langle \mathbf{w} \rangle)}{\|\mathbf{w} - \langle \mathbf{w} \rangle\|}.$$

- Constant sectional curvature 4.

- Intrinsic mean of a probability distribution Q exists if its support contained in ball of radius atmost $\frac{\pi}{8}$.

Future work

To come up with more general results on the existence (uniqueness) of Intrinsic means.

Few Definitions

1. Geodesic: $\ddot{\gamma} = 0$. Locally length minimizing curves.
   eg. Great Circles on S^n, Straight lines in $\mathbb{R}^n$.

2. Exponential map: $p \in M$, $V \in T_pM$; $exp_pV = \gamma(1)$, where γ is a geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$.

3. Cut locus C(p): Let γ be a geodesic with $\gamma(0) = p$. Let t_0 be the supremum of all t for which γ is length minimizing on $[0, t]$. Then $\gamma(t_0)$ is the cut point of p along γ. $C(p)$ is the set of all cut points of p along all geodesics.
   eg. $C(p) = \{-p\}$ on S^n.

3. Sectional Curvature: For a curve γ, its sectional curvature is $\pm|\ddot{\gamma}(t)|$, $+$ if the curve is pointing towards N and $-$ if it is pointing away from N, where N is a chosen normal field along γ.

   For a $2d$ manifold, chose a basis (X, Y) for T_pM. Then sectional curvature at p is $\frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - <X,Y>^2}$.

   For a d dimensional manifold, consider the $2d$ submanifold swept out by geodesics with initial velocities lying in a $2d$ subspace, Π of T_pM. That is the sectional curvature at p associated with Π, $K(\Pi)$.