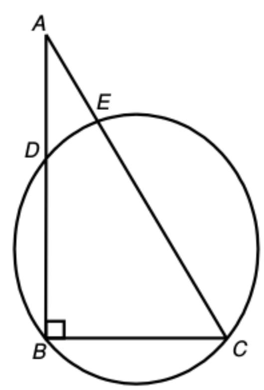
(1) Let the sequence $\{a_n\}_{n\geq 1}$ be defined by

$$a_n = \tan(n\theta),$$

where $\tan(\theta) = 2$. Show that for all n, a_n is a rational number which can be written with an odd denominator.

(2) Consider a circle of radius 6 as given in the diagram below. Let B, C, D and E be points on the circle such that BD and CE, when extended, intersect at A. If AD and AE have length 5 and 4 respectively, and DBC is a right angle, then show that the length of BC is $\frac{12+9\sqrt{15}}{5}$.



(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function given by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ e^{(x^{10}-1)} + (x-1)^2 \sin\left(\frac{1}{x-1}\right) & \text{if } x \neq 1. \end{cases}$$

(a) Find $f'(1)$.

(b) Evaluate $\displaystyle\lim_{u\to\infty} \left[100\,u - u\sum_{k=1}^{100} f\left(1+\frac{k}{u}\right)\right]$.

(4) Let S be the square formed by the four vertices $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Let the region R be the set of points inside S which are closer to the centre than to any of the four sides. Find the area of the region R.

P.T.O.

(5) Let $g : \mathbb{N} \to \mathbb{N}$ with $g(n)$ being the product of the digits of n.
   (a) Prove that $g(n) \leq n$ for all $n \in \mathbb{N}$.
   (b) Find all $n \in \mathbb{N}$, for which $n^2 - 12n + 36 = g(n)$.

(6) Let p_1, p_2, p_3 be primes with $p_2 \neq p_3$, such that $4 + p_1p_2$ and $4 + p_1p_3$ are perfect squares. Find all possible values of p_1, p_2, p_3.

(7) Let $A = \{1, 2, \ldots, n\}$. For a permutation $P = (P(1), P(2), \cdots, P(n))$ of the elements of A, let $P(1)$ denote the first element of P. Find the number of all such permutations P so that for all $i, j \in A$:
- if $i < j < P(1)$, then j appears before i in P; and
- if $P(1) < i < j$, then i appears before j in P.

(8) Let k, n and r be positive integers.
   (a) Let $Q(x) = x^k + a_1x^{k+1} + \cdots + a_nx^{k+n}$ be a polynomial with real coefficients. Show that the function $\frac{Q(x)}{x^k}$ is strictly positive for all real x satisfying
$$0 < |x| < \frac{1}{1 + \sum_{i=1}^{n} |a_i|}.$$
   (b) Let $P(x) = b_0 + b_1x + \cdots + b_rx^r$ be a non-zero polynomial with real coefficients. Let m be the smallest number such that $b_m \neq 0$. Prove that the graph of $y = P(x)$ cuts the x-axis at the origin (i.e. P changes sign at $x = 0$) if and only if m is an odd integer.