1. Let i be a root of the equation $x^2 + 1 = 0$ and let ω be a root of the equation $x^2 + x + 1 = 0$. Construct a polynomial

$$f(x) = a_0 + \ a_1 x + \ \ \ldots + \ a_n x^n$$

where $a_0, a_1, \ldots, a_n$ are all integers such that $f(i + \omega) = 0$.

2. Let a be a fixed real number. Consider the equation

$$(x + 2)^2 (x + 7)^2 + a = 0, \ \ x \in \mathrm{R},$$

where R is the set of real numbers. For what values of a, will the equation have exactly one double-root?

3. Let A and B be variable points on x-axis and y-axis respectively such that the line segment AB is in the first quadrant and of a fixed length $2d$. Let C be the mid-point of AB and P be a point such that

   (a) P and the origin are on the opposite sides of AB and,

   (b) PC is a line segment of length d which is perpendicular to AB.

   Find the locus of P.

4. Let a real-valued sequence $\{x_n\}_{n \geq 1}$ be such that

$$\lim_{n \to \infty} n x_n = 0.$$

   Find all possible real values of t such that $\lim_{n \to \infty} x_n (\log\, n)^t = 0$.

5. Prove that the largest pentagon (in terms of area) that can be inscribed in a circle of radius 1 is regular (i.e., has equal sides).

6. Prove that the family of curves

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} \ = \ 1$$

   satisfies

$$\frac{dy}{dx}(a^2 - b^2) = (x + y\frac{dy}{dx}) \ (x\frac{dy}{dx} - y).$$

7. Consider a right-angled triangle with integer-valued sides $a < b < c$ where a, b, c are pairwise co-prime. Let $d = c - b$. Suppose d divides a. Then

   (a) Prove that $d \leq 2$.

   (b) Find all such triangles (i.e. all possible triplets a, b, c) with perimeter less than 100.

8. A finite sequence of numbers $(a_1, \ldots, a_n)$ is said to be *alternating* if

$$a_1 > a_2, \quad a_2 < a_3, \quad a_3 > a_4, \quad a_4 < a_5, \ldots.$$

$$\text{or} \quad a_1 < a_2, \quad a_2 > a_3, \quad a_3 < a_4, \quad a_4 > a_5, \ldots.$$

   How many alternating sequences of length 5, with *distinct* numbers $a_1, \ldots, a_5$ can be formed such that $a_i \in \{1, 2, \ldots, 20\}$ for $i = 1, \ldots, 5$?