MTA

Notations :

$\mathbb{Z}$ = Set of integers.

$\mathbb{N}$ = Set of natural numbers = $\{n \in \mathbb{Z} : n \geq 1\}$.

$\mathbb{Q}$ = Set of rationals.

$\mathbb{R}$ = Set of real numbers.

$\mathbb{C}$ = Set of complex numbers.

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1. Prove that the following limit exists and find the limit:

$$\lim_{n\to\infty} \Big((1 + \frac{1}{2} + \cdots + \frac{1}{n}) - \ln n \Big).$$

2. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then for any $r_1, r_2, \cdots, r_n \in f[a, b]$, prove that there exists $x \in [a, b]$ such that $f(x) = \frac{r_1+r_2+\cdots+r_n}{n}$.

3. Suppose $y(x) = x^2$ is a solution of $y'' + P(x)y' + Q(x)y = 0$ on $(0, 1)$ where P and Q are continuous functions on $(0, 1)$. Can both P and Q be bounded functions. Justify your answer.

4. For each $\alpha > 0$, find all pairs of $(x_0, y_0) \in \mathbb{R}^2$ such that the following initial value problem has a unique solution in the neighbourhood of (x_0, y_0) -

$$y' = y^\alpha \ ; \ y(x_0) = y_0.$$

5. Let $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\| \leq 1\}$, and let

$$f(x) = \inf\{\|x - y\| : y \in B\}, \qquad \forall \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

If $F(x) = \max\{1 - f(x), 0\}, x = (x_1, x_2) \in \mathbb{R}^2$, then prove that

$$\lim_{n\to\infty} \int\int_{\mathbb{R}^2} F^n(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 = \pi.$$

6. Let $f : X \to Y$ be a function from a metric space (X, d_1) to a compact metric space (Y, d_2). Let $G_f := \{(x, y) : y = f(x)\} \subset X \times Y$ denote the graph of f. Show that f is continous iff G_f is closed in $X \times Y$. The metric d on $X \times Y$ is the product metric which is defined as $d((x_1, y_1), (x_2, y_2)) := \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2}$.

7. Let $f : \mathbb{R} \to [0, \infty)$ be a Borel measurable function. Show that

$$\sum_{n=1}^{\infty} m(\{f \geq n\}) \leq \int f \mathrm{d}m \leq \sum_{n=1}^{\infty} m(\{f > n\}),$$

where m denotes the Lebesgue measure.

8. Let (a_n) be a sequence of real numbers and m be the Lebesgue measure. Suppose $\frac{1}{n}\sum_{k=1}^{n} f(a_k) \to \int_{\mathbb{R}} f dm$ for all Lebesgue integrable functions f on $\mathbb{R}$. Prove that (a_n) is dense in $\mathbb{R}$.

9. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function i.e., analytic everywhere in $\mathbb{C}$. Suppose

$$\lim_{|z| \to \infty} \frac{f(z)}{z} = 0.$$

Prove that f is a constant function.

10. What are the holomorphic functions f on an open connected subset $\Omega \subset \mathbb{C}$ such that $g \colon \Omega \to \mathbb{C}$ defined by $g(z) = Re(z)f(z)$ is also holomorphic.