MTB
2019

Notation

$\mathbb{Z}$ = the set of integers

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{R}$ = the set of real numbers

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{C}$ = the set of complex numbers

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(1) Let A be a diagonalisable $m \times m$ matrix with entries from $\mathbb{C}$. Define $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ assuming A^0 is the identity matrix. Prove that $\det(\exp(A)) = e^{\mathrm{Tr}(A)}$.

(2) Let $H = \{1 + 4k : k \in \mathbb{Z},\ k \geq 0\}$. An element $x \in H$ is called *H-prime* if $x \neq 1$ and x cannot be written as product of two strictly smaller elements of H.

(i) Show that $xy \in H$ for all $x, y \in H$.

(ii) Prove that every $x \in H$ greater than 1 can be factored as a product of H-primes but unique factorisation does not hold.

(3) Find an ideal I in $A = \frac{\mathbb{Z}[X]}{(X^4 + X^2 + 1)}$ such that $\frac{A}{I}$ is a finite field with 25 elements.

(4) (a) Suppose A and B are closed subsets of a topological space such that $A \cap B$ and $A \cup B$ are connected. Prove that A and B are connected.

(b) Demonstrate that the conclusion may not hold if the assumption of A and B being closed subsets, is dropped.

(5) Examine whether there is a polynomial $f(X) \in \mathbb{R}[X]$ such that $\frac{\mathbb{R}[X]}{(f(X))}$ is isomorphic **as rings** to the product ring $\mathbb{C} \times \mathbb{C}$.

(6) Let $f(x), g(x) \in \mathbb{Z}[X]$ with $f(x) = \sum_{j=0}^{n} a_j x^j$ and $b(x) = \sum_{j=0}^{n} b_j x^j$. For $m \in \mathbb{N}$, we say $f \equiv g \mod m$ if $a_j = b_j \mod m$ for $0 \le j \le n$. For an odd prime $p > 0$, let

$$f(x) = x^{p-1} - 1 \quad \text{and} \quad g(x) = (x-1)(x-2)\cdots(x-p+1) \; .$$

Prove that

(i) the polynomial $f(x) - g(x)$ has degree $p - 2$, and

(ii) $f \equiv g \mod p$.

(7) Prove that the following two groups are isomorphic:

(a) $\mathbb{Z}[X]$, the group of polynomials with integer coefficients under addition, and

(b) $\mathbb{Q}_{>0}$, the group of positive rational numbers under multiplication.

Hint: Fundamental theorem of Arithmetic.

(8) Prove that there cannot be any topological space X such that $\mathbb{R}$ is homeomorphic to $X \times X$ with the product topology.

(9) Let $A = \dfrac{\mathbb{C}[X, Y]}{(X^2 + Y^2 - 1)}$, and x, y denote the images of X, Y in A respectively. If $u = x + iy$, then prove that

(a) u is a unit in A, and

(b) $u - i$ generates a maximal ideal of A.

(10) Let A be a real symmetric $m \times m$ matrix with m distinct eigenvalues and $v_1, \ldots, v_m$ be the corresponding eigenvectors. Let C be an $m \times m$ matrix satisfying $\langle Cv_j \, , v_j \rangle = 0$ for $1 \le j \le m$. Prove that there exists an $m \times m$ matrix X such that $AX - XA = C$.