- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{Q}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.

1. Let $V_1, V_2 \subseteq V$ be two subspaces of a vector space V and $b_1, b_2 \in V$. Show that the affine subspaces $W_1 = V_1 + b_1$ and $W_2 = V_2 + b_2$ have a nonempty intersection if and only if $\pi(b_1) = \pi(b_2)$, where $\pi : V \to V/(V_1 + V_2)$ is the quotient map.

2. (a) Let L_1 and L_2 be fields and $L_1 \subseteq L_2$. Let $A = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}\}$ be a set of k many vectors in $L_1^n \subseteq L_2^n$. Show that the linear independence of A in L_1^n over L_1 implies the linear independence of A in L_2^n over L_2. (Hint: Consider the case $k = n$.)

   (b) Let $\alpha \in \mathbb{C}$ be a root of the polynomial $X^4 + 1 \in \mathbb{Q}[X]$. Consider the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(\alpha)$.

      i. With complete justification determine the degree of this extension.

      ii. Find three fields K_1, K_2 , K_3 such that $\mathbb{Q} \subset K_i \subset \mathbb{Q}(\alpha)$ for $i = 1, 2, 3$.

3. Let n be a positive integer and $\phi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_n$ be the homomorphism defined by $\phi(1) = (n, \bar{1})$, where $\bar{1}$ denotes the set of integers congruent to 1 mod n. Prove that $(\mathbb{Z} \oplus \mathbb{Z}_n)/\mathrm{Im}(\phi)$ is cyclic and find its order.

4. Let X be a set containing at least two elements and G be a group with identity element e. Suppose that there is a map $\alpha : G \times X \to X$. Let us write $\alpha(g, x)$ as $g \cdot x$. Assume that the following properties are satisfied by α:

   - $e \cdot x = x$ for all $x \in X$;
   - $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$;

- given $(x_1, x_2) \in X \times X$ and $(y_1, y_2) \in X \times X$ with $x_1 \neq x_2$, $y_1 \neq y_2$, there is a $g \in G$ such that $g \cdot x_1 = y_1$ and $g \cdot x_2 = y_2$. (The case x_i is same as some y_j for $i, j = 1, 2$ is allowed).

For $x \in X$, define

$$G_x := \{g \in G \, : g \cdot x = x\}.$$

Prove the following:

(a) Given any pair $a, b \in X$, there is a $g \in G$ such that $g \cdot a = b$.

(b) $G_x \neq G$ for any $x \in X$.

(c) Let $x \in X$ and $g \in G \setminus G_x$. Then $G = G_x \cup G_x \, g \, G_x$.

(d) For any $x \in X$, there is no proper subgroup of G properly containing G_x.

5. Let R be an integral domain. For $a, b \in R$, by "*a divides b*" we mean that there is an $x \in R$ such that $ax = b$. We recall the following definitions.

   An element $d \in R$ is a gcd of $a, b \in R$ if:

   - d divides both a and b, and
   - if $c \in R$ divides both a and b, then c divides d.

   An element $l \in R$ is an lcm of $a, b \in R$ if:

   - both a and b divide l, and
   - if both a and b divide $m \in R$, then l divides m.

   Assuming that any two elements in R have an lcm, prove the following.

   (a) Any two elements in R have a gcd.

   (b) Any irreducible element in R is prime.

6. Prove that the space $[0, 1]$ is not homeomorphic to a product $X \times Y$ for two subsets X and Y of $\mathbb{R}$, where neither X nor Y is a single point.

7. Let L_1 be the line $y = 1$, L_{-1} be the line $y = -1$, and for every $n \geq 1$, R_n be the rectangle

$$R_n = \left(\{-n, n\} \times [-1 + \tfrac{1}{n}, 1 - \tfrac{1}{n}]\right) \cup \left([-n, n] \times \{-1 + \tfrac{1}{n}, 1 - \tfrac{1}{n}\}\right).$$

Define

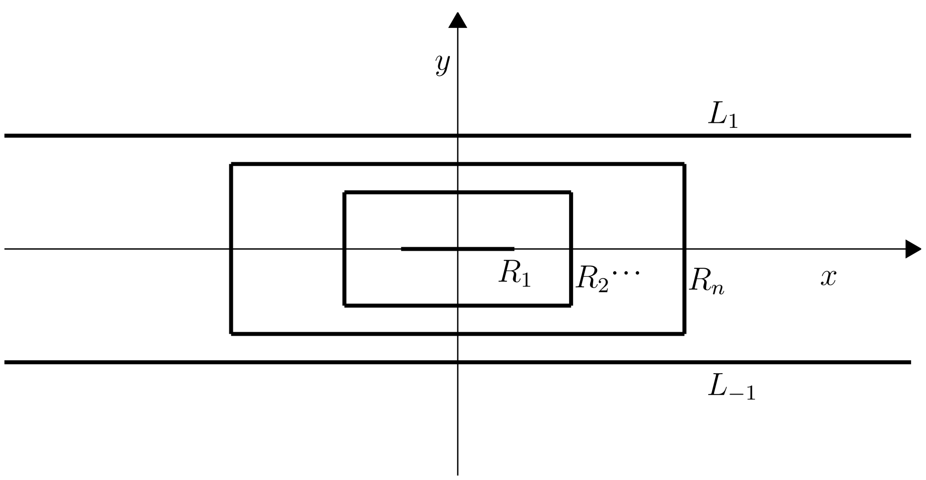
$$X = L_1 \cup L_{-1} \cup \Big(\bigcup_{n\geq 1} R_n\Big),$$

which is equipped with the subspace topology from $\mathbb{R}^2$ (see the figure below).

(a) Find the connected components of X.

(b) Find points $a, b \in X$ in different connected components of X such that there is no disconnection of $X = A \cup B$ with $a \in A$ and $b \in B$. [$X = A \cup B$ is a disconnection means A and B are disjoint open sets.]

Justify your answers in (a) and (b).



8. Let $|\cdot|_2 : \mathbb{Q} \to [0, \infty)$ be the map given by

$$|x|_2 = \begin{cases} 0, & \text{if } x = 0, \\ 2^{-r}, & \text{if } x = 2^r \frac{m}{n}, \quad m, n, r \in \mathbb{Z} \text{ with } m, n \text{ odd.} \end{cases}$$

(a) Show that $|x + y|_2 \leq \max\{|x|_2, |y|_2\}$ for all $x, y \in \mathbb{Q}$ and that equality holds if $|x|_2 \neq |y|_2$.

(b) Using this or otherwise show that for $n > 1$, $\sum_{k=1}^{n} \frac{1}{k}$ is not an integer.