Notation

- $\mathbb{N}$ = natural numbers • $\mathbb{Z}$ = integers • $\mathbb{Q}$ = rational numbers
- $\mathbb{R}$ = real numbers • $\mathbb{C}$ = complex numbers

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1. Let A be a square matrix with real entries such that $A^{2022} = 0$.

   (a) Compute trace(A^2).

   (b) If further it is given that A is symmetric, then identify all possible matrices that A can be.

2. Suppose A and B are two 9×9 matrices with real entries such that rank$(A) = 3$ and rank$(B) = 5$. Show that there exists a nonzero column vector $\mathbf{x} \in \mathbb{R}^9$ such that $A\mathbf{x} = B\mathbf{x} = \mathbf{0}$.

3. Let X, Y, Z be Banach spaces, $T : X \to Y$ be a linear map, $S : Y \to Z$ be a bounded injective linear map such that $S \circ T$ is a bounded map. Show that T is bounded.

4. Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{v_n\}_{n \geq 1}$. Let $\{q_1, q_2, \cdots\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$. Consider the dense subspace $D = \text{span}\{v_n \mid n \geq 1\}$ in $\mathcal{H}$ and the map $T : D \to \mathcal{H}$ defined by

$$T\left(\sum_{k \geq 1} a_k v_k\right) = \sum_{k \geq 1} a_k q_k v_k$$

   where a_k's are in $\mathbb{C}$ such that only finitely many are nonzero. Show that

   (a) T extends to a bounded linear operator $\widetilde{T} : \mathcal{H} \to \mathcal{H}$.

   (b) The extension $\widetilde{T} : \mathcal{H} \to \mathcal{H}$ is not compact.

5. A test for divisibility by 7 goes as follows. To check if a number n is divisible by 7, the double of its unit digit is subtracted from the number $\widehat{n}$ formed by deleting the unit digit of n. The original number n is declared to be divisible by 7 if and only if this difference is divisible by 7. (For example, 112 is divisible by 7 because $11 - (2 \times 2) = 7$ but 423 is not divisible by 7 because $42 - (2 \times 3) = 36$.) Show that this is a legitimate divisibility test for 7.

6. Let $\mathcal{P}$ be a regular polygon with n sides inscribed in a circle.

   (a) Suppose there exists a rectangle, all of whose vertices are among the vertices of $\mathcal{P}$. Then show that n must be an even number.

   (b) Suppose $n = 2m$. Calculate the number of rectangles having all their vertices among the vertices of $\mathcal{P}$.

7. Let $p(t), q(t) \in \mathbb{Z}[t]$ be non-constant polynomials that have no common roots in $\mathbb{C}$. Prove that any ideal $I \subset \mathbb{Z}[t]$ containing $p(t)$ and $q(t)$ must contain a nonzero integer.

8. Prove or disprove:

   (a) There exist only finitely many homomorphisms from the additive group $(\mathbb{Q}, +)$ to the multiplicative group $(\mathbb{C}^*, \cdot)$.

   (b) There exists only one homomorphism from the additive group $(\mathbb{Q}, +)$ to the additive group $(\mathbb{Z}, +)$.

9. Let G be a group of order 77^2. Show that $G \cong H \times K$ where H and K are groups of orders 49 and 121, respectively.

10. A PCR-test given to a person is known to be 90% reliable when the person has the disease and 99% reliable when the person does not have the disease. Suppose Sheela was selected from a group of persons of which only 5% had the disease and the PCR-test indicates that she has the disease, what is the probability that she indeed has the disease?