Part I

Answer all questions.

1. Find the value of x for which the following equation is satisfied

$$\begin{vmatrix} (1+x)^2 & (1+2x)^2 & (1+3x)^2 \\ (2+x)^2 & (2+2x)^2 & (2+3x)^2 \\ (3+x)^2 & (3+2x)^2 & (3+3x)^2 \end{vmatrix} = -648x.$$

[5]

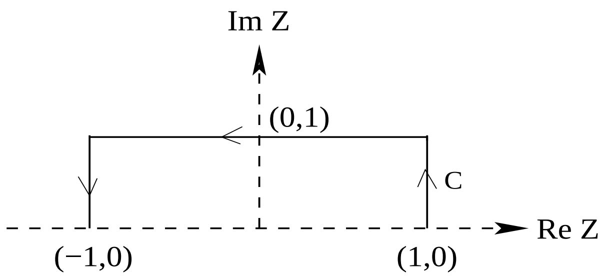
2. Find the particular solution of the differential equation

$$3\,x\,\frac{dy}{dx} - y = \ln(x) + 1, \quad x > 0,$$

satisfying the condition $y(1) = -2$.

[5]

3. Find the value of the integral $\int_C z^2\,e^z\,dz$, where C is an open contour in the complex Z-plane as shown in the figure below.



[5]

4. A mass m when suspended from a light spring causes an extension α. If a mass M is added to m, find the time period and amplitude of the oscillation.

[2+3]

5. An observer in a rocket moving parallel to a long platform measures the length of the platform to be 300 meters while the length of the platform measured by a stationary observer is 500 meters. Find the time taken by the rocket to cover the platform with respect to the platform observer.

[5]

6. An electromagnetic wave propagating in free space is described by

$$\mathbf{E}(x, y, z, t) = \frac{V_0}{a} \cos\left(\frac{3x}{a} - \frac{4y}{a} - \omega t\right) \hat{\mathbf{z}}$$

where V_0 and a are constants and the other symbols have their usual meaning. Find the wavelength and period of the wave. Also determine the direction of propagation.

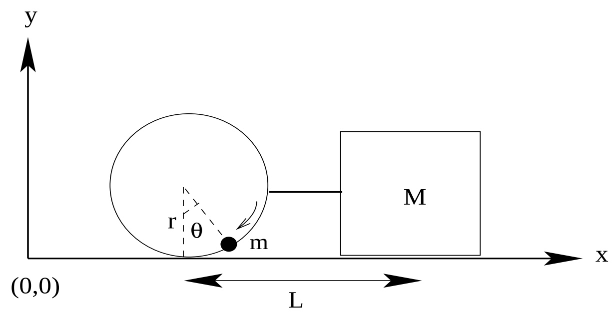
[(2+2)+1]

Part II

Physics

Answer any five questions.

1. (a) A block of mass M is rigidly connected to a massless vertical circular track of radius r on a frictionless table.The distance between the centre of the circular track and centre of the mass M is L. A particle of mass m is restricted to move on the vertical circular track without friction. Please see the figure below.



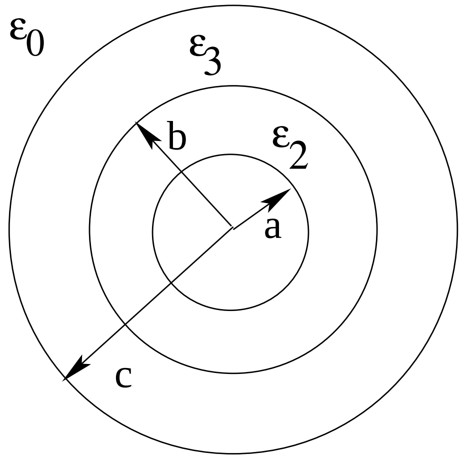
   i. Set up the Lagrangian using coordinates θ and x.
   ii. Obtain the equations of motion.

   (b) A particle of mass m is bound by a linear potential $U = kr$ where k is a constant.

   i. For what energy and angular momentum will the orbit be a circle of radius r about the origin?
   ii. What is the frequency of this circular motion?
   iii. If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillation?

[(4+3)+(2+2+3)]

2. (a) A particle of mass m and charge q in the presence of electric field $\vec{E}$ and magnetic field $\vec{B}$ experiences the Lorentz force $\vec{F} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$ where $\vec{v}$ is the particle velocity. If $\vec{E} = -\vec{\nabla}\phi(x, y, z)$ is independent of time, show that the particle energy $\frac{1}{2}m\vec{v}.\vec{v} + q\phi$ is also independent of time.



(b) Consider two infinitely long cylinders, as shown in the figure above. The inner cylinder of radius a is a conductor with linear charge density $\lambda_1 > 0$. The second cylinder with inner radius b and outer radius c consists of a material with permittivity ε_3 and is uniformly charged with line charge density $\lambda_3 < 0$ ($\lambda_1 > |\lambda_3|$). The space between the two cylinders (i.e., $a < r < b$) is filled with a medium of permittivity ε_2. The medium outside the outer cylinder possesses the permittivity ε_0. Compute the potential difference between a point at $|\vec{r}| = 2c$ and the center of the inner cylinder.

[5+9]

3. (a) Consider an operator $\hat{A} = \frac{d^2}{d\phi^2}$, acting on periodic functions of period 2π i.e., $f(\phi + 2\pi) = f(\phi)$, where ϕ is the azimuthal angle in polar coordinates ($0 \le \phi \le 2\pi$).

   i. Check whether $\hat{A}$ is Hermitian.

   ii. Determine the eigenvalues and eigenfunctions of $\hat{A}$ and specify the degenerate eigenvalues.

   (b) Consider a charged particle in one-dimensional harmonic oscillator potential. Suppose we turn on a weak electric field $\vec{E}$, so that the potential energy is shifted by an amount $-qEx$. Calculate the first and second order corrections to the energies.

[(2+5)+(3+4)]

4. (a) A one-dimensional free particle wave function is given by

$$\psi(x,t) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\left(p_x x - \frac{p_x^2}{2m}t\right)\right] \phi(p_x)\, dp_x,$$

where the symbols have their usual meaning. Show that the expectation values of the position $\langle x \rangle$ and momentum $\langle p_x \rangle$ can be related by the expression

$$\langle x \rangle = \langle x \rangle_{t=t_0} + \frac{\langle p_x \rangle}{m}(t - t_0).$$

(b) In relativistic quantum mechanics, fermions are described by Dirac equation with the Hamiltonian

$$\hat{H} = -i\hbar c \overrightarrow{\alpha}.\overrightarrow{\nabla} + \beta m_0 c^2,$$

where $\overrightarrow{\alpha} \equiv (\alpha_1, \alpha_2, \alpha_3)$ along three spatial directions. The α_i's and β are Hermitian, traceless matrices with eigenvalues ± 1. The rest mass of the fermion is m_0 and c is the speed of light in vacuum.

i. Derive the equation of continuity.

ii. Show that the total probability is conserved.

[6+(6+2)]

5. (a) Calculate the number of ways in which N identical bosons can be distributed in two energy levels.

(b) A column of a liquid contains suspended metal particles of radius r_0 which are in thermal equilibrium at temperature T. Let the densities of the liquid and metal be ρ and ρ_0 respectively. If there are N particles per unit volume at a given height H, find the number of particles per unit volume at a position h units above H.

(c) Consider a gas consisting of N noninteracting spins (S=1) in a magnetic field $\vec{B}$, such that the Hamiltonian is given by $B\hat{S}_z$. Using canonical ensemble, find the average magnetization of the gas.

[2+5+7]

6. (a) In tight-binding approximation the dispersion relation of electrons in a 3D lattice is given by

$$E_k = \alpha\cos(k_x a) + \beta\cos(k_y a) + \gamma\cos(k_z a)$$

where a is the lattice constant and α, β, γ are constants with dimension of energy. Find the effective mass tensor at the corner of the first Brillouin zone $\left(\frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}\right)$.

(b) A type-II superconductor is placed in a small magnetic field, which is then slowly increased till the field starts penetrating the superconductor. The strength of the field at this point is $\frac{2}{\pi} \times 10^5$ Gauss. Assuming the flux-quantum (fluxoid) to be 2×10^{-7} Gauss-cm^2, find the penetrating depth of this superconductor.

(c) Consider a free electron gas in three-dimensional space.

i. Write the Hamiltonian of the system.

ii. Find the quantized wave vectors and allowed eigenenergies for the cases of periodic and hard-wall boundary conditions.

iii. For the above two boundary conditions, calculate the degeneracy factor of the single-particle ground state energy. Set the system temperature at absolute zero.

[3+5+(1+4+1)]

7. (a) The orbit of a particle of mass m under a central force with centre at the origin is described by the following trajectory

$$r = a(1 + cos\theta), \quad a \,\text{is a constant.}$$

i. Show that the force is inversely proportional to r^4.

ii. Write the Hamiltonian for the particle in polar coordinates.

(b) Consider the following interaction:

$$\pi^0 \to \gamma + \gamma.$$

Here a neutral π^0 meson decays into two highly energetic photons. The rest mass of the π^0 meson is 135 MeV/c^2.

i. Find the energy of each photon when the π^0 particle decays at rest.

ii. Suppose the π^0 meson decays while in motion and it has total energy 426 MeV in the laboratory system. If the photons now move apart at arbitrary angles with respect to the direction of the moving meson, what are the maximum and minimum energies the photons can attain?

[(4+3)+(2+5)]

8. (a) Consider an interacting quantum field theory of two scalar fields ϕ and ψ in (3+1) dimensions with the interaction term of the form $\lambda\phi\psi^3$, λ being the coupling constant. Assuming the mass of ϕ as m and ψ as massless, write down the action of the theory. Draw possible Feynman diagrams for the following processes:

   i. One ϕ particle and one ψ particle going to two ψ particles.

   ii. Two ϕ particles going to two ψ particles.

   In each case label the momenta in the propagators for some arbitrary initial and final momentum configurations.

   (b) A π^+ meson is created in a high-energy collision of a primary cosmic ray particle in the earth's atmosphere 200 km above the sea level. It descends vertically at a speed $0.99c$ (c is the speed of light in vacuum) and it disintegrates, in its proper frame, 2.5×10^{-8} seconds after its creation. Determine the altitude above the sea level at which the disintegration occurs.

[(1+3+3)+7]