Part I

Answer <u>all</u> questions

1. Determine all the square roots of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

[5]

2. Find the Fourier series representing $f(x) = x, \quad 0 < x < 2\pi$.

[5]

3. Evaluate the integral

$$\oint_C \frac{z^2}{e^z + 1}\, dz$$

where C is the circle $|z| = 4$.

[5]

4. Calculate the change in entropy if x grams of water at temperature $T_1\,^\circ$C is added to y grams of water at temperature $T_2\,^\circ$C ($T_1 > T_2$).

[5]

5. A pendulum is suspended in a lift. When the lift is stationary, the period of oscillation of the pendulum is T_0. Determine the period of oscillation of the pendulum when the lift begins to accelerate downwards with an acceleration $\frac{3}{4}g$, where g is the acceleration due to gravity.

[5]

6. Consider a single electron atom with orbital angular momentum $L = \sqrt{2}\hbar$. What are the possible values of a measurement of L_z, the z-component of L?

[5]

Part II

Physics

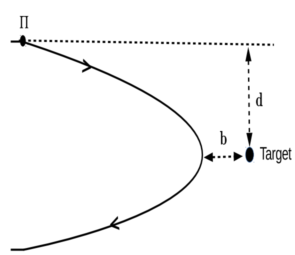
Answer <u>any five</u> questions

1. (a) A particle of mass m is sliding on a smooth surface along a path that satisfies $r^2 = az$ where a is a constant. Using cylindrical coordinates r, ϕ, z where $x = r\ cos\phi, y = r\ sin\phi, z = z$:

    i. Set up the Lagrangian and find the equations of motion.

    ii. Identify the cyclic coordinate.

   (b) In a non-relativistic system, a charged pion (with charge $+q$ for π^+ or $-q$ for π^-) has kinetic energy T and it is moving towards a massive target nucleus with charge Q. The pion is considered to hit the nucleus if its distance is b from the nucleus (see figure). The collision cross-section is given by $\Sigma = \pi d^2$, where d is the impact parameter.



Show that the cross-sections, Σ^+ and Σ^- for π^+ and π^- respectively are given by

$$\Sigma^+ = \frac{\pi b^2(T-V)}{T}, \qquad \Sigma^- = \frac{\pi b^2(T+V)}{T}$$

where $V = \frac{qQ}{b}$ is the Coulomb potential.

• Impact parameter d is defined as the length of the perpendicular drawn from the target (nucleus) to the line of motion that the pion would have taken if there was no interaction.

[(5+1)+8]

2. (a) A particle of mass m is placed in a finite spherical well of radius a with the following potential:

$$V(r) = \begin{cases} -V_0 & \text{if } r \leq a \\ 0 & \text{if } r > a \end{cases}$$

i. Solving the radial equation with $\ell = 0$ find the ground state wave function.

ii. Show that there is no bound state if

$$V_0 < \frac{\pi^2 \hbar^2}{8ma^2}$$

(b) The Hamiltonian for a spin-$\frac{1}{2}$ particle of mass m with charge $+e$ in an external magnetic field $\vec{B}$ is

$$H = -\frac{ge}{2mc}\, \vec{s}.\vec{B}$$

where the symbols have their usual meaning.

i. Derive the expression for $\frac{d\vec{s}}{dt}$.

ii. Assuming $\vec{B} = B\hat{y}$, find $s_z(t)$ in terms of the given quantities.

[(4+3)+(4+3)]

3. (a) Two synchronized clocks A and B are at rest in an inertial reference frame. The distance between them is L. Another clock X is moving with a velocity $\frac{3}{5}c$ along the line joining A and B, c being the velocity of light in vacuum. Both the clocks A and X read zero when X passes A. When X reaches the mid point of the line joining A and B, what are the readings of clocks A and B with respect to the inertial frame attached with X?

(b) Consider the following state of a quantum harmonic oscillator

$$|\psi\rangle = c_0|0\rangle + c_k|k\rangle$$

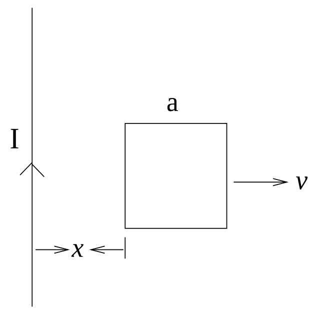
where $|0\rangle$ and $|k\rangle$ are the energy eigenstates. The non-zero real coefficients c_0 and c_k satisfy $c_0^2 + c_k^2 = 1$. Find the allowed values of k for which

$$\langle\psi|\frac{1}{2}m\omega^2\hat{x}^2|\psi\rangle = \langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle$$

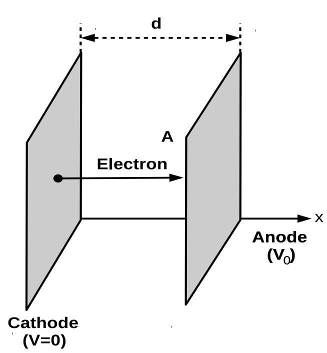
where the symbols have their usual meaning.

[6+8]

4. (a) A square frame with side a and a long wire carrying a current I are located in the same plane as shown in the figure below. The frame translates to the right with a constant velocity v. Find the e.m.f. induced in the frame as a function of x.



(b) Suppose there are two parallel plate electrodes of area A, at voltages 0 and V_0 respectively, separated by a distance d (see the figure below). The dimensions of the plates are much larger than the separation between them. With an unlimited supply of electrons at rest to the lower potential electrode (placed at $x = 0$), a steady current I flows between the plates.



i. Write the Poisson's equation for the region between the plates.
ii. What is the speed of the electrons at point x, where the potential is $V(x)$?
iii. Show that $\frac{d^2V}{dx^2} = \beta V^{-1/2}$ and find the constant β.

[6+(1+2+5)]

5. (a) A system with two degrees of freedom is described by the Hamiltonian

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2$$

where q_i, p_j obey canonical Poisson brackets and a, b are numerical constants. Show that

$$F_1 = \frac{p_1 - a q_1}{q_2} \ , \qquad F_2 = q_1 q_2$$

are constants of motion.

(b) A K-meson of rest energy 494 MeV decays into a muon of rest energy 106 MeV and a neutrino of zero rest energy. Find the kinetic energies of muon and neutrino in the rest frame of K-meson in which K-meson decays at rest.

(c) State the conservation laws that are violated for the following processes:

i. $\nu_\mu + n \longrightarrow e^- + p$

ii. $p + \bar{p} \longrightarrow \Lambda^0 + \Lambda^0$

[(3+3)+6+(1+1)]

6. (a) Suppose the density of states of a free electron gas in three dimensions gets increased by eight times.

i. Explain on physical grounds whether the Fermi temperature of the system increases or decreases.

ii. Find out the factor by which it increases or decreases.

(b) Consider a system of N non-interacting spins each with a magnetic moment of magnitude μ. The system is placed in an external uniform magnetic field $\vec{B}$.

i. Write down the Hamiltonian of the system

ii. Calculate the magnetization per spin at temperature T.

[(2+4)+(3+5)]

7. Consider a one-dimensional tight-binding periodic lattice with lattice constant a, on-site energy ϵ and nearest-neighbor hopping strength t.

i. Determine the energy dispersion (viz, E-k) relation.

ii. Find the wave vector k in terms of lattice constant and total number of lattice sites for an N-site lattice under periodic and finite boundary conditions.

iii. Calculate the energy band width for the periodic case.

[6+(3+3)+2]

8. (a) Consider a free scalar field.

   i. Derive an expression for the Hamiltonian in terms of creation and annihilation operators.

   ii. What is the energy of the vacuum?

   (b) Consider a field theory where a Dirac electron ψ interacts with a charged scalar ϕ and a neutral scalar η.

   i. Give examples of simplest possible interaction terms in both cases, maintaining gauge invariance.

   ii. Draw Feynman diagrams for the processes

$$\psi^- \, \phi^+ \;\longrightarrow\; \psi^- \, \phi^+$$

   and

$$\psi^- \, \eta \longrightarrow \psi^- \, \eta.$$

[(4+3)+(4+3)]