Part-I

Mathematical and Logical Reasoning

Answer __all__ questions. Each question carries 5 marks.

1. Two water drops of equal radius, each with a charge of 3×10^{-9} Coulomb and having a surface potential of 500 Volts, form a single drop. Calculate the surface potential of the new drop so formed. [5]

2. An Olympic diver of mass m begins his descent from a height h above the water surface with zero initial velocity. Considering the viscous force is $(-k)$ times velocity, and the gravitational force balances the buoyancy inside water, find out the expression for velocity of the diver as a function of vertical depth x inside water. [5]

3. The growth rate of a bacterial population is proportional to the population itself. The population becomes three times of its initial value in one hour. After how many hours will the population reach 100 times of its initial value? [5]

4. Find the work done in moving a particle once around an ellipse in the x-y plane, centered at the origin. The semi-major (along x axis) and semi-minor (along y axis) axes are 4 and 3, respectively, and the force field is given by

$$\mathbf{F} = (3x - 4y + 2z)\hat{\mathbf{x}} + (4x + 2y - 3z^2)\hat{\mathbf{y}} + (2xz - 4y^2 + z^3)\hat{\mathbf{z}}.$$

[5]

5. Evaluate the integral

$$\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx.$$

[5]

6. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. If $u = 3x - 2xy$, then find v and express $f(z)$ in terms of z. [5]

Part-II

Physics

Answer any five questions. Each question carries 14 *marks.*

1. (a) A magnetic monopole of strength g, placed at the origin, generates a magnetic field $g\frac{\mathbf{r}}{r^3}$, where $\mathbf{r}$ is the position vector measured from the origin.

   i. Consider a particle of mass m and electric charge e moving in this field. Write down the equation of motion of the particle.

   ii. Show that an effective angular momentum of the particle given by $\mathbf{J} = m\mathbf{r} \times \dot{\mathbf{r}} - eg\frac{\mathbf{r}}{r}$ is conserved.

   (b) Consider a Hamiltonian

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)$$

   where q, p constitute a canonical pair of degrees of freedom.

   i. Find the equation of motion for q that involves only q and its time derivatives.

   ii. Find a set of canonical co-ordinates Q, P in terms of q, p such that the Hamiltonian $H(Q, P)$ corresponds to the form of a harmonic oscillator.

[(3+4)+(3+4)]

2. (a) A large mass M, moving at relativistic speed v, collides with a small mass m which is initially at rest. The two masses stick together after collision. Considering $M \gg m$, find out the approximate mass of the resulting object up to first order in $\frac{m}{M}$.

   (b) Consider the Lagrangian of a particle, in one-dimension, given by

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)$$

   where m, k and γ are constants.

i. Discuss the possible motions of the particle depending on the conditions on the parameters m, k and γ.

ii. Derive the motion of the particle in terms of a new variable S defined by $S = e^{(\gamma t/2)q}$.

[6+(4+4)]

3. (a) A particle of mass m is in the *asymmetric* potential well

$$V(x) = \begin{cases} V_1, & x < 0 \\ 0, & 0 < x < a \\ V_2, & x > a \end{cases}$$

where $V_1 > V_2 > 0$. Find out its discrete energy levels.

(b) A quantum system with two orthonormal states, say $|0\rangle$ and $|1\rangle$, is described by the Hamiltonian $\hat{H} = |0\rangle\langle 1| + |1\rangle\langle 0|$. If at $t = 0$, the expectation value of the operator $\hat{A} = 3|0\rangle\langle 0| - |1\rangle\langle 1|$ is $\langle \hat{A} \rangle = -1$, what is the initial state $|\psi(0)\rangle$? What is the smallest time $t > 0$ such that $|\psi(t)\rangle = |0\rangle$?

(c) A quantum harmonic oscillator, in the usual notation, is described by the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{m\omega^2}{2} x^2.$$

Consider a trial wavefunction of the form $\psi(x) = \frac{A}{x^2+B^2}$, where A and B are constants. Determine A and B for the best bound on its ground state energy.

[5+(2+2)+5]

4. (a) Consider a rigid body with moment of inertia I_x rotating freely in a plane with its rotating axis making an angle ϕ with x-axis. The Hamiltonian operator for this system is given by

$$\hat{H} = -\frac{\hbar^2}{2I_x} \frac{d^2}{d\phi^2}.$$

i. Find out the energy eigenvalues and eigenfunctions.

ii. Suppose at time $t = 0$ the rigid rotator is described by a wave packet $\psi(\phi, 0) = A \sin^2 \phi$. Find out $\psi(\phi, t)$ for any arbitrary time.

(b) Consider an electron of mass m in an infinite cubic potential well

$$V(x,y,z) = \begin{cases} 0 & \text{if } 0 < \{x,y,z\} < L \\ \infty & \text{otherwise.} \end{cases}$$

Now the following perturbation is switched on

$$H_p = V_0 L^3 \delta\left(x - \frac{L}{4}\right)\,\delta\left(y - \frac{3L}{4}\right)\,\delta\left(z - \frac{L}{4}\right)$$

where V_0 is a constant. Using first order perturbation theory, calculate the energy of the ground state.

[(4+5)+5]

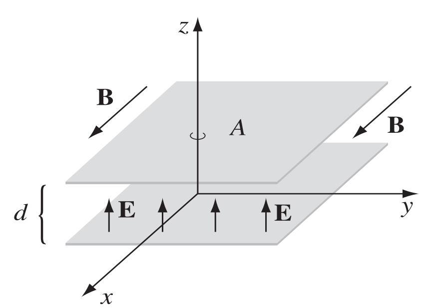
5. (a) Suppose a magnetic monopole q_m passes through a resistanceless loop of wire with self-inductance L. What current is induced in the loop?

(b) A thick spherical shell of inner radius a and outer radius b is made of dielectric material with a frozen-in polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}}$$

where k is a constant and r is the distance from the centre. Find the bound charge and use Gauss's law to calculate the electric field.

(c) A charged parallel plate capacitor having area A with uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$ is placed in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$ (see Figure).



i. Find the electromagnetic momentum in the space between the plates.

ii. Suppose a resistive wire is connected between the plates along the z axis, so that the capacitor slowly discharges. Calculate the total impulse delivered to the system during this discharge.

[6+4+(1+3)]

6. (a) The dispersion relation of a gas of spin-$\frac{1}{2}$ fermions in two dimensions is $E = \hbar v k$, where E is the energy, k is the wavevector, and v is a constant. If the Fermi energy at zero temperature is E_F, then find the number of particles per unit area in terms of v and E_F.

   (b) A $2\,\text{mm} \times 2\,\text{mm}$ two-dimensional metal film contains 4×10^{12} electrons. Determine the magnitude of the Fermi wavevector of the system in the free electron approximation, considering spin degeneracy.

   (c) The electrical conductivity of copper is 95% of that of silver, whereas the electron density in silver is 70% of that of copper. Using Drude's model, find the ratio of their mean collision times.

   (d) Optical excitation of an intrinsic germanium creates an average density of 10^{12} conduction electrons per cm^3 in the material of liquid nitrogen temperature. At this temperature, the electron and hole mobilities are equal and given by 0.5×10^4 cm^2/volt sec. If 100 volts is applied across a $1\,\text{cm}$ cube of crystal under these conditions, how much current will be observed?

[4+3+3+4]

7. (a) A two-level system has energies 0 and ϵ, with degeneracies 1 and 3, respectively. Calculate the partition function Z and the temperature at which the probability of finding the system in the lower energy level is equal to half. Assume that the system obeys classical statistics.

   (b) Consider a cubical box of side L in which 10 identical spinless particles, each of mass m, need to be accommodated.

Determine the lowest energy of the system when the particles obey

i. BE statistics
ii. FD statistics.

[(2+3)+(2+7)]

8. (a) Consider a reaction $p + p \rightarrow x + K^+ + K^+$, where x is an unknown particle.

   i. What are the values of electric charge, strangeness and baryon number of x?
   ii. Let the mass of K^+ and proton be 0.494 GeV and 0.938 GeV respectively, and the target protons are at rest. What is the minimum momentum of the incident proton required for the reaction to take place, if the mass of x is 2.15 GeV?

   (b) Consider an interacting quantum field theory of electron field ψ with mass m, and a complex scalar field ϕ with mass M. The interaction vertex can contain at most two ψ fields, two ϕ fields and one derivative.

   i. Write down the most general form of the Lagrangian for the system.
   ii. Draw the interaction vertices that will appear in Feynman diagrams.

[(3+4)+(4+3)]

Part-III

Mathematics

Answer any five questions. Each question carries 14 marks.

1. (a) Consider the ring

$$\mathbb{Z}[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$$

with the usual addition and multiplication of complex numbers. Let

$$S = \{a + ib \in \mathbb{Z}[i] \mid a, b \in \mathbb{Z},\ b \text{ is even }\}.$$

Show that S is a subring of $\mathbb{Z}[i]$, but not an ideal of $\mathbb{Z}[i]$.

(b) Let S be a non-empty set with an associative binary operation that is left and right cancellation (i.e., $xy = xz \Rightarrow y = z$, and $yx = zx \Rightarrow y = z$). Assume that for $a \in S$, the set $\{a^n \mid n = 1, 2, 3, \ldots\}$ is finite. Prove that S is a group.

[(3+3)+8]

2. (a) A steady 2-dimensional fluid flow is described by

$$V_r = -u\Big(1 - \frac{a^2}{r^2}\Big)\cos\theta,\ \ V_\theta = u\Big(1 + \frac{a^2}{r^2}\Big)\sin\theta.$$

Find the acceleration of a fluid particle at the point $r = 2a,\ \theta = \frac{\pi}{2}$.

(b) A fluid flows at a constant discharge Q through a convergent pipe of length L having inlet and outlet radii R_1 and R_2 ($R_1 > R_2$), respectively. Assuming the velocity to be axial and uniform at any cross section, find out the acceleration at the exit.

(c) Find the series solution of the following equation using Frobenius method near $x = 0$

$$(x - x^2)\frac{d^2y}{dx^2} + (1 - 5x)\frac{dy}{dx} - 4y = 0.$$

[4+4+6]

3. (a) Let P be a $n \times n$ real orthogonal matrix with $det(P) = -1$. Discuss whether the inverse of the matrix $(P + I_n)$ exists or not. Justify your answer.

   (b) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x^2 + 6x - 5$ is both one-to-one and onto.

   (c) Let $f(x, y) = y + x \sin(\frac{1}{y})$, if $y \neq 0$ and $f(x, 0) = 0$. Discuss the continuity of the function $f(x, y)$ at $(0, 0)$.

[5+4+5]

4. (a) Suppose W is a subspace of $\mathbb{R}^n$ of dimension d. Prove that

$$\left| \left\{ (x_1, x_2, \ldots, x_n) \in W \ : \ x_i \in \{0, 1\} \right\} \right| \leq 2^d,$$

   where for any finite set A, $|A|$ denotes its number of elements.

   (b) Let $[x]$ denote the integer nearest to x. Draw the graph of the function

$$y = |x - [x]|, \ 0 \leq x \leq 4.$$

   Find all the points $x \in [0, 4]$ where the function is not differentiable. Justify your answer.

[6+8]

5. (a) Classify and reduce the equation

$$\frac{\partial^2 u}{\partial x^2} - 2 \sin x \frac{\partial^2 u}{\partial x \partial y} - \cos^2 x \frac{\partial^2 u}{\partial y^2} - \cos x \frac{\partial u}{\partial y} = 0$$

   to a canonical form and hence solve it.

   (b) Let E be the set of all differentiable functions $x(t)$ whose derivatives are continuous on the bounded closed interval $a \leq t \leq b$. Prove that

$$\rho(x, y) = |x(a) - y(a)| + \sup \left\{ \left| \frac{dx}{dt} - \frac{dy}{dt} \right| : t \in [a, b] \right\}$$

   is a metric on E.

[8+6]

6. (a) Find all the equilibrium points and discuss the corresponding linear stability analysis of the system

$$\frac{dx}{dt} = y - x - g(x)$$
$$\frac{dy}{dt} = x - y$$

with $g(x) = \alpha x + \beta\Big(|x+1| - |x-1|\Big)$, where α and β are parameters.

(b) Find the Taylor series of the function

$$f(z) = \frac{z^2 + z}{(1-z)^2}$$

at $z = -1$ and its radius of convergence.

[9+(3+2)]

7. (a) Let $\{Z_n\}$ be a stochastic process. Define a filtration $\{\mathbb{F}_n\}$. Describe when $\{Z_n\}$ may be called a martingale with respect to this filtration.

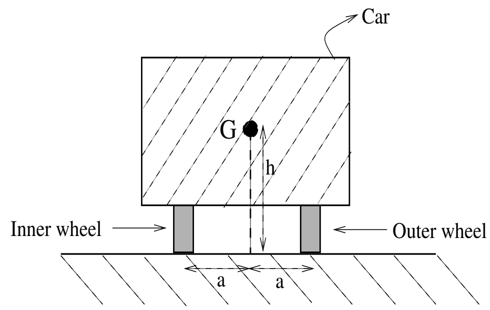
(b) Let $\{X_n\}$ be a sequence of independent $N(0,2)$ random variables. Let $Z_0 = 0$ and $Z_n = X_1 + X_2 + ... + X_n (n \geq 1)$. Let $M_n = \max_{0 \leq k \leq n} |Z_k|$.

i. Show that Z_n is a martingale with respect to an appropriate filtration by describing one such filtration.
ii. Show that $P(M_n \geq 2\sqrt{n}) \leq \frac{1}{2}$.
iii. Further, show that $E(M_n^2) \leq 8n$.

[(2+2)+(3+4+3)]

8. A car of mass m is traveling on a horizontal road (see figure) with a speed v so that the center of mass G of the car describes a circle of radius r. The separation between the inner and outer wheels is $2a$. The height of the center of mass G of the car above the ground is h.

(a) Show the different forces on the car.
(b) Show the moments about G when the car is moving.



(c) Find the maximum value of the velocity of the car above which the car will overturn.
(Hint: Find the condition for which the normal reaction on the inner wheel will vanish, i.e., the inner wheel will lose contact with the ground.)

[3+4+7]