Part-I

Mathematical and Logical Reasoning

Answer all questions. Each question carries 5 marks.

1. If $y = x$ is one solution of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0,$$

then find the general solution.

[5]

2. Find $\iint_R x^2 y^2 \, dx \, dy$ over a region R bounded by $x \geq 0, y \geq 0$ and $x^2 + y^2 \leq 1$.

[5]

3. Let A be a Hermitian matrix. Show that the matrix $A + i\mathbb{I}$ is invertible and $(A + i\mathbb{I})^{-1}(A - i\mathbb{I})$ is a unitary matrix.

[5]

4. Find the probability of the occurrence of a number that is odd or less than 5 when a fair die is rolled.

[5]

5. A uniform chain of length l and mass m overhangs a smooth table with its two third part lying on the table. Find the kinetic energy of the chain as it completely slips off the table.

[5]

6. A small mass initially at rest, starts sliding downwards from the top of a frictionless sphere of radius R. At what point does it loose contact with the sphere?

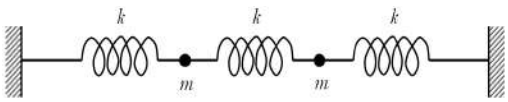
[5]

Part-II

Physics

Answer any five questions. Each question carries 14 marks.

1. Two small balls of mass m are suspended from two rigid supports by three springs, of equal spring constant k (see figure). Both balls can move only in the plane of the figure, that is only up-down and sideways. In the unstretched condition, each spring is of length $a/2$.



   (a) Set up Lagrangian of the system.

   (b) Consider small oscillations of the balls about the equilibrium configuration as shown in the figure. Write the approximate form of the Lagrangian.

   (c) Derive the equations of motion.

   (d) Derive expressions for any two of the normal modes.

[3+4+3+4]

2. (a) Two identical bodies have internal energy $U = NCT$ each, with a constant C. The values of N and C are the same for each body. The initial temperatures of the bodies are T_1 and T_2, and they are used as a source of work by connecting them to a Carnot heat engine and bringing them to a common final temperature T_f.

   i. What is the final temperature T_f?

   ii. What is the work delivered?

   (b) Consider an ideal gas of stable vector mesons (massive particles with spin 1) with magnetic moment μ. Determine the magnetic susceptibility per unit volume in a weak magnetic field. Assume the number of particles per unit volume of the gas is n.

[(2+5)+7]

3. Two concentric metal spheres of radii a and b $(a < b)$ are separated by a medium that has dielectric constant ϵ and conductivity σ. At time $t = 0$ an electric charge q is suddenly placed on the inner sphere.

   (a) Calculate the total current through the medium as a function of time.

   (b) Determine the Joule heat produced by this current.

   (c) Find the electrostatic energy due to discharging.

[6+4+4]

4. (a) A train with proper length L moves at speed $c/2$ (c being the velocity of light in vacuum) with respect to the ground. Inside the train, a ball is thrown from the back to the front, at speed $c/3$ with respect to the train.

      i. How much time does this take and what distance does the ball cover,

         A. in the rest frame of the train?

         B. with respect to the earth?

         C. in the rest frame of the ball?

      ii. Verify that the invariant interval is indeed the same in all three frames.

      iii. Find the relation between the times elapsed in the rest frame of the ball and the earth.

   (b) Show that the following transformation is canonical:

$$\begin{aligned} Q &= \left(e^{-2q} - p^2\right)^{1/2} \\ P &= \cos^{-1}\left(pe^q\right). \end{aligned}$$

[((2+2+2)+3+2)+3]

5. (a) Assuming that each atom of copper contributes one free electron, determine the drift velocity of free electrons in a copper conductor, having cross-sectional area $10^{-4}\,\mathrm{m}^2$, carrying a current of $200\mathrm{A}$. (Atomic weight of copper $= 63.5$, Density of copper $= 8.94 \times 10^3\,\mathrm{kgm}^{-3}$)

   (b) The spacing of the planes in a crystal is 1.2Å and the angle for the first order reflection is $30°$. If the spacing of the crystal planes changes by $\pm 0.1\text{Å}$, then find the spread in energy in the diffracted beam.

   (c) If the band gap of an alloy semiconductor is $1.98\,\mathrm{eV}$, then find the wavelength of radiation that is emitted when electrons and holes in the material recombine directly.

[6+5+3]

6. Consider a quantum linear harmonic oscillator with Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

where the symbols have their usual meanings.

   (a) Derive the equation of motion for the expectation value of the position $\langle x \rangle$ and show that it oscillates, similar to the classical oscillator, as

$$\langle x \rangle_t = \langle x \rangle_{t=t_0} \cos\omega t + \frac{\langle p_x \rangle_{t=t_0}}{m\omega} \sin\omega t.$$

   (b) Using the commutation relations between x, p_x and their products, show that

$$\langle x^2 \rangle_t + \frac{\langle p_x^2 \rangle_t}{m^2\omega^2} = \langle x^2 \rangle_{t=0} + \frac{\langle p_x^2 \rangle_{t=0}}{m^2\omega^2}.$$

   (c) Hence show that

$$\begin{aligned}(\Delta x)_t^2 &= (\Delta x)_{t=t_0}^2 \cos^2\omega t + \frac{(\Delta p_x)_{t=t_0}^2}{m^2\omega^2}\sin^2\omega t \\ &\quad + \left[\frac{1}{2}\langle x p_x + p_x x \rangle_{t=t_0} - \langle x \rangle_{t=t_0} \langle p_x \rangle_{t=t_0}\right] \frac{\sin 2\omega t}{m\omega}.\end{aligned}$$

[6+4+4]

7. (a) Consider the one-dimensional wave function

$$\psi(x) = A\,(x/x_0)^n\, e^{-x/x_0},$$

where A, n and x_0 are constants.
Using the Schrödinger's equation, find the potential $V(x)$ and energy E for which the wave function is an eigenfunction. (Assume that for $x \to \infty$, $V(x) \to 0$).

(b) Consider a two-dimensional oscillator

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right).$$

The Hamiltonian is given in units for which $\hbar = m = \omega = 1$.

i. What are the lowest 3 eigenfunctions and the corresponding eigenenergies?

ii. Next consider a perturbation to the Hamiltonian

$$V = \frac{1}{2}\epsilon xy\left(x^2 + y^2\right), \qquad (\epsilon << 1).$$

Using the first order perturbation theory, find the effect of V on the energies of the states calculated in part (i).

[4+((2+2)+6)]

8. (a) Consider an interacting model of two relativistic scalar fields $\phi(x)$ and $\psi(x)$ having mass parameters m and M respectively with interaction terms $\lambda\phi^2\psi^2 + \sigma\phi\psi^3$, where λ, σ are coupling constants.

i. Derive the equations of motion.

ii. Draw lowest order Feynman diagrams for the following processes, with the coupling constants involved and label the incoming and outgoing particle momenta keeping in mind momentum conservation principle:

A. $\phi + \phi \to \psi + \psi$

B. $\phi + \phi \to \phi + \phi$

iii. For process (ii) write down the amplitude in momentum space. (Detailed numerical factors may be ignored.)

(b) Check which of the following processes are forbidden and why?

i. $\pi^+ \to e^+ + \nu_e$

ii. $K^+ \to \pi^+ + \pi^+ + \pi^0$

[(4+(2+2)+2)+(2+2)]

Part-III

Mathematics

Answer any five questions. Each question carries 14 marks.

1. (a) Consider the initial value problem

$$\frac{dy}{dx} = -\sqrt{|1-y^2(x)|}, \quad y(0) = 1.$$

i. Does it admit a unique solution in the neighborhood of $y = 1$? Justify your answer.

ii. Find the solution(s) of the differential equation, if exist.

(b) Consider the system,

$$\frac{dy}{dx} = x - rx(1-x),$$

where x is a state variable and $r \in \mathbb{R}$ is a parameter.

i. Discuss the linear stability analysis of all equilibrium points.

ii. Sketch the bifurcation diagram of equilibrium points x^* vs. r.

iii. Calculate r_c, the critical parameter value at which the bifurcation occurs and identify the type of bifurcation.

[(3+2)+(3+4+2)]

2. (a) Let f be analytic in a region G and suppose that f is not identically zero. Let $G_0 = G \setminus \{z : f(z) = 0\}$ and define $h : G_0 \to \mathbb{R}$ by $h(z) = \log|f(z)|$. Show that

$$\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} = \frac{f'}{f} \text{ on } G_0.$$

(b) Let f be an entire function and suppose there is a constant M and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > \xi > 0$. Show that f is a polynomial of degree less than or equal to n.

[7+7]

3. (a) Find the Bessel function $J_p(x)$ of first kind of order p by solving the differential equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0, \quad p \text{ is a constant.}$$

Also, show that $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$.

(b) Can you find a Markov chain of finite number of states with at least one null recurrent state? Justify your answer.

[(6+3)+5]

4. (a) Let $\mathbb{P} = \{2, 3, 5, 7, 11, \ldots\}$ be the set of all prime numbers. Define (explicitly and as simple as you can) an operation $*$ on $\mathbb{P}$ such that $(\mathbb{P}, *)$ satisfies the first three axioms for a group, that is, closure, associativity, and existence of an identity element.

(b) How many *zero divisor* are there in $\mathbb{Z}_{2021}$?

(c) Consider $\mathbb{R}^2$ with the usual addition on it. If the multiplication is defined as $c.(x, y) = (ax + by, ax - by)$ for $c = a + ib \in \mathbb{C}$, will $(\mathbb{R}^2, +, .)$ be a vector space over $\mathbb{C}$? Justify your answer.

[4+5+5]

5. (a) Let A be a complex square matrix of order m such that $A^n = \mathbb{I}_m$ for some positive integer n, where $\mathbb{I}_m$ denotes the $m \times m$ identity matrix.

   i. Show that A is diagonalizable.

   ii. Give an example of $A \neq \mathbb{I}_2$ such that $A^3 = \mathbb{I}_2$.

(b) Show that $x^4 + 1$ is reducible over $\mathbb{Z}_3$.

[(6+3)+5]

6. (a) Does $f(y) - f(x) = f'(z)(y - x)$ necessarily true for a differentiable map $f : \mathbb{R} \to \mathbb{R}^2$ and $x, y \in \mathbb{R}$, where $z \in \{(1-t)x + ty : t \in [0, 1]\}$? Justify your answer.

(b) Prove or disprove both the necessary and sufficient part of the following statement:

f is a Lebesgue measurable function if and only if $f^{-1}(c)$ is Lebesgue measurable for each number $c \in \mathbb{R}$.

[5+9]

7. (a) A velocity field is given by $\vec{U} = (1 + At + Bt^2)\,\hat{\imath} + x\hat{\jmath}$, where A, B are constants, and, $\hat{\imath}$, $\hat{\jmath}$ are two orthogonal unit vectors.

   i. Find the equation of the streamline at $t = t_0$ passing through the point (x_0, y_0).

   ii. Obtain the path line of a fluid element which passes through (x_0, y_0) at $t = t_0$.

   iii. Show that the streamline and path line coincide, if the flow is steady.

   (b) A two-dimensional velocity field is given by

$$u = -\frac{Ky}{x^2 + y^2}, \quad v = \frac{Kx}{x^2 + y^2},$$

   where K is a constant.

   i. Does the field satisfy incompressible continuity?

   ii. Transform these velocities into polar components v_r and v_θ.

   iii. What might the flow represent?

[(3+3+2)+(2+2+2)]

8. (a) A particle is constrained to move on the plane curve $xy = 1$ under the gravity g. Taking y-axis vertical, the Lagrange's equation is obtained as $x^2\ddot{x} + \frac{d}{dt}F(x, \dot{x}) - g = 0$. Find the value of the function $F(x, \dot{x})$.

   (b) As seen from Earth, two spaceships A and B are approaching along perpendicular directions. If A is observed by a stationary Earth observer to have velocity $u_y = -0.9c$ and B to have velocity $u_x = +0.9c$, determine the speed of ship A as measured by the pilot of ship B. Here c denotes the speed of light in vacuum.

[8+6]