Part-I

Mathematical and Logical Reasoning

Answer all questions. Each question carries 5 marks.

1. Let a, b are integers such that $x = 2i + \sqrt{3}$ is a solution of the equation

$$x^5 - x^4 + 2x^3 - 2x^2 + ax - b = 0.$$

Find the values of a and b.

[2+3]

2. Show that there does not exist any 4×4 real symmetric orthogonal matrix with all diagonal elements zero.

[5]

3. Find a homogeneous linear differential equation with constant coefficients which is satisfied by the function $y = 1 + x + \cos^3 x$.

[5]

4. Evaluate $\displaystyle\iint_S \operatorname{curl} \vec{F}.\, \hat{n}\, dS$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = a^2$ and $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$.

[5]

5. From the six letters P, Q, R, S, T and U, three letters are chosen at random with replacement. What is the probability that either the word "QPS" or the word "UPS" can be formed from the chosen letters?

[5]

6. Evaluate

$$\int_0^\infty \frac{\cos mx}{x^2 + 1}\, dx.$$

[5]

Part-II

Physics

Answer any five questions. Each question carries 14 marks.

1. (a) Determine the path on which an object, in the absence of friction, will fall from one given point from rest to another point in the shortest possible time under the effect of the gravity.

   (b) State Hamilton's principle for a conservative system, and use it to find the equation of motion

   $$mf = -\frac{\partial V}{\partial x}$$

   for a particle of mass m moving with acceleration f in a potential V.

   (c) Using Lagrange's equations of motion, determine the motion of a mass m, sliding down through an inclined plane of angle α. Assume zero of potential energy at the top of the plane, and ignore the friction.

   (d) Find the Lagrangian for an electrical circuit comprising an inductance L and capacitance C. The capacitor is charged to q coulombs and the current flowing in the circuit is i amperes.

   [5+3+4+2]

2. (a) The path of a particle moving under a central force, is given by the spiral

   $$r = A\exp(\alpha\theta),$$

   where A and α are constants. The angular momentum L and the total energy E of the particle are known.

   i. Find the form of the effective potential $V(r)$ in the above scenario.

   ii. Analyze the stability of this motion from the plot of the effective potential $V(r)$ versus r.

(b) The Hamiltonian of a three-dimensional harmonic oscillator is

$$H = \frac{1}{2}\sum_{i=1}^{3}(p_i^2 + \mu q_i^2)$$

where μ is a constant.

i. Show that the following quantities are constants of motion.

■ $F_1 = q_2p_3 - q_3p_2$,

■ $G_1 = \mu q_1 cos(\mu t) - p_1 sin(\mu t)$.

ii. From the symmetry of the system, write down the expressions of other similar conserved quantities F_2, F_3, G_2, G_3.

[(4+3)+((2+2)+3)]

3. (a) A perfect gas is expanded from $10\,\mathrm{m}^3$ to $20\,\mathrm{m}^3$ at a constant pressure of $10^5\,\mathrm{N/m}^2$. The temperature before the expansion was 100°C. Find the temperature after expansion.

(b) The energy dispersion of the conduction band of a one-dimensional metal is given by

$$E_c(k_x) = E_g + E_1 \sin^2(k_x a/2)$$

with $\hbar k_x$ is the crystal momentum, a is the lattice constant, and E_g and E_1 are constants.

i. Find the effective masses of an electron at $k_x = 0$ and π/a.

ii. Derive an expression for the group velocity v_g of an electron.

iii. Plot v_g as a function of k_x in the first Brillouin zone.

[5+(4+2+3)]

4. The electrons in a metallic solid may be considered to be a three-dimensional free electron gas. For this case:

(a) Obtain the allowed values of k, and sketch the appropriate Fermi sphere in k-space. (Use periodic boundary conditions with length L).

(b) Find the maximum value of k for a system with N_e number of electrons, and hence an expression for the Fermi energy at $T = 0\,\mathrm{K}$.

(c) Using a simple argument show that the contribution the electrons make to the specific heat is proportional to the temperature T.

(d) Find the Lande g-factor for the $^3P_1$ level of an atom.

[(3+2)+(3+2)+2+2]

5. A spherically symmetric charge distribution is given by

$$\rho(r) = \begin{cases} \rho_0(1 - \frac{r^2}{a^2}) & \text{for} \quad 0 \leq r < a \\ 0 & \text{for} \quad r > a. \end{cases}$$

(a) Determine the total charge.

(b) Find the electric field intensity E and potential V, both inside and outside the charge distribution.

[4+(5+5)]

6. (a) Consider a particle in the potential $V(x) = \infty$ for $x \leq 0$ and $x \geq L$ and $V(x) = 0$ for $0 \leq x \leq L$. The wave function of the particle at $t = 0$ is the following

$$|\psi(x)\rangle = \frac{2}{\sqrt{L}} \sin\left(\frac{5\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right).$$

i. Find the wave function at $t = T$.

ii. If the energy of the particle is measured at time $t = T$, what are the possible measurement results? Find the corresponding probabilities.

(b) Consider the following state of a quantum harmonic oscillator

$$|\psi\rangle = c_0|0\rangle + c_k|k\rangle$$

where $|0\rangle$ is the ground state and $|k\rangle$ is the k-th energy eigenstates of the harmonic oscillator with $k > 0$. c_0, c_k are both non-zero real with $c_0^2 + c_k^2 = 1$.

Find the allowed values of k for which

$$\langle\psi|\frac{1}{2}m\omega^2\hat{x}^2|\psi\rangle = \langle\psi|\frac{\hat{p}^2}{2m}|\psi\rangle.$$

[(2+(3+2))+7]

7. (a) A hydrogen atom, initially in its ground state, is placed in a time-dependent electric field (pointing along z-axis) which is given by

$$\overrightarrow{E}(t) = E_0\tau\frac{\overrightarrow{k}}{\tau^2 + t^2}.$$

Here τ is a constant having the dimension of time. Calculate the probability that the atom will be found in $2p$ state at $t = +\infty$.

(b) In an inertial frame, two events have the space-time coordinates $\{x_1, y, z, t_1\}$ and $\{x_2, y, z, t_2\}$, respectively. Let $x_2 - x_1 = 7c(t_2 - t_1)$ (c represents the velocity of light in vacuum). Consider another inertial frame which moves along x-axis with velocity u with respect to the first one. Find the value of u for which the events are simultaneous in the later frame.

[7+7]

8. (a) Consider the following relation between energy and momentum for a quantum particle moving in an electromagnetic field

$$(E - eV)^2 = \left(\overrightarrow{p} - e\overrightarrow{A}\right)^2 c^2 + m_0^2 c^4$$

where m_0 and c are constants, called rest mass and speed of light in vacuum respectively, and V and $\overrightarrow{A}$ are respectively the scalar and vector potentials of an electromagnetic field.

i. By directly quantizing energy and momentum, find the equation of motion for a particle with wave function $\psi(x, t)$ moving under such a field.

ii. Find out the probability current density and check if it is positive definite.

(b) Consider a massive complex scalar field.

i. Find the equations of motion.

ii. Find the Noether charge current.

iii. Now introduce a self-interaction term that is of fourth power in the complex scalar field and again find the Noether charge current.

[(4+3)+(2+2+3)]

Part-III

Mathematics

Answer any five questions. Each question carries 14 marks.

1. (a) Let G be a group. Show that

$$x^3 = x,\ \forall x \in G \ \Rightarrow\ G \text{ is abelian.}$$

Now let G be a ring instead. Justify if the statement,

$$x^3 = x,\ \forall x \in G \ \Rightarrow\ G \text{ is abelian,}$$

still holds.

(b) Let $2 \leq n \in \mathbb{Z}$ and $S \subset \mathbb{R}^n$ be such that any (set of) n members of S are linearly independent. How big the set S could be (that is, what is the maximum possible cardinality of S)? Justify your answer.

[(2+5)+(1+6)]

2. (a) Find one ordinary point and series solution of the following differential equation

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0.$$

(b) Solve the initial-boundary value problem,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad 0 < x < \pi, t > 0 \\ u(x,0) = 1, \qquad & \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < \pi \\ u(0,t) = 0, \qquad & u(\pi,t) = 0, \quad t \geq 0. \end{aligned}$$

[(1+6)+7]

3. (a) Find the equilibrium points and discuss the linear stability analysis of the system of equations

$$\frac{dx}{dt} = x(x+y-1) \quad \text{and} \quad \frac{dy}{dt} = y(x-y+\beta),$$

where β is a real parameter. Also find the bifurcation points.

(b) Consider X and Y are independent random variables, each with probability density function (pdf)

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

where $\lambda > 0$. Find the probability density function of $W = Y - X^2$.

[(7+2)+5]

4. (a) Let $x \in \mathbb{C}^n$. What are the eigenvalues of

$$\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}?$$

Here x^* denotes the conjugate transpose of x.

(b) Let A, B are complex matrices of order $m \times n$ and $n \times m$ respectively. Show that if $\mathbb{I}_m - AB$ is nonsingular, then $\mathbb{I}_n - BA$ must be nonsingular as well.

(c) Find the integral

$$\int_C \tan z \, dz,$$

where C is the circle of radius 8 centered at the origin.

[3+4+7]

5. (a) Discuss the differentiability of

$$f(x, y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$$

at $(0, 0)$.

(b) Is the series with n-th partial sum $\left\{ \dfrac{nx}{1 + n^2x^2} \right\}_{n=1}^{\infty}$ uniformly convergent on $[-1, 1]$? Justify your answer.

(c) Let d and d' be metrics on a non-empty set X. Prove or disprove the following statements:

i. $d_1(x, y) = \max\{d(x, y), d'(x, y)\}$ for all $x, y \in X$ necessarily a metric on X.

ii. $d_2(x, y) = \min\{d(x, y), d'(x, y)\}$ for all $x, y \in X$ necessarily a metric on X.

[5+5+4]

6. (a) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function satisfying $u(x, y) \leq x$ for all $z = x + iy$. Then prove or disprove the following statement:

   $f(z)$ is a polynomial of degree at most one.

   (b) Define f on $[0, 1]$ by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ x^3, & \text{if } x \text{ is irrational.} \end{cases}$$

   Discuss the Riemann integrability of f on $[0, 1]$.

   (c) Let $f_n \in L^1(0, 1) \cap L^2(0, 1)$ for all $n \in \mathbb{N}$, where $L^p(E)$ denotes L^p space defined over E. Prove or disprove the following statements:

   i. If $\|f_n\|_1 \to 0$, then $\|f_n\|_2 \to 0$.

   ii. If $\|f_n\|_2 \to 0$, then $\|f_n\|_1 \to 0$.

[5+4+5]

7. (a) The velocity components of two-dimensional incompressible flow are given by $u = 2xy$ and $v = a^2 + x^2 - y^2$, where u and v are the velocity components in x and y-directions, respectively. Evaluate the expression of the velocity potential function.

   (b) Find the conditions on a and b so that the following transformation $((q, p) \to (Q, P))$ is canonical.

$$q = a\sqrt{2P}\sin Q, \qquad p = b\sqrt{2P}\cos Q.$$

   (c) For the system with Hamiltonian

$$H = q_1p_1 - 3q_2p_2 - q_1^2 + 2q_2^2,$$

   which of the following statements are correct?

i. $\dot{p}_1 = -p_1 + 2q$

ii. $\dot{p}_2 = 3p_2 - 2q_2$

iii. $q_1 = Ae^t$

iv. $q_2 = Be^{-3t}$

Justify your answer.

[5+5+4]

8. (a) Suppose, a two-dimensional flow is described in the Lagrangian system by the following equations

$$x = x_0 e^{-kt} + y_0\left(1 - e^{-2kt}\right) \quad \text{and} \quad y = y_0 e^{kt}.$$

i. Find the equation of path line of the particle.

ii. Determine the velocity components u and v in x and y-directions, respectively in the Eulerian system.

(b) A spherical pendulum consists of a point mass m tied by a string of length l to a fixed point, so that it is constrained to move on a spherical surface as shown in Figure 1. Let $\dot{\varphi}$ be the angular velocity of m about z-axis.

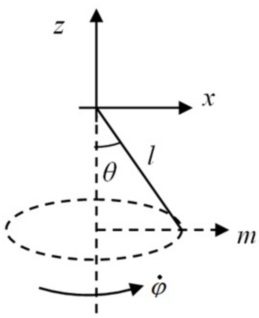

Figure 1: A spherical pendulum under motion.

With what angular velocity will it move on a circle, with the string making a constant angle θ_0 with the vertical?

[(4+4)+6]