QEA
2019

1. Consider the set, A, of all rectangles of area 100 square units. Let $a \in A$ be a rectangle which minimizes the sum of the width and twice the length among all members of A. Find the length and the width of every such a.

2. Prove or disprove the following statement: the function $x \ln_e x$ defined on the positive orthant $\mathbb{R}_{++}$ is convex.

3. Consider a non-negative random variable X with the distribution function F. Suppose that F is strictly increasing and differentiable. Prove or disprove the following statement: the expected value of X can be represented as
$$\int_{t=0}^{t=1} F^{-1}(t)dt.$$

4. A class has 100 students. Let a_i, $1 \leq i \leq 100$, denote the number of friends the i-th student has in the class. For each $0 \leq j \leq 99$, let c_j be the number of students having at least j friends. Derive the value of $\sum_{i=1}^{100} a_i - \sum_{j=1}^{99} c_j$.

5. Consider the 3×3 matrix
$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$
Find all the eigenvalues of A, A^2, and $A + 5I$, where I is the identity matrix of order 3×3.

6. Suppose that the probability of getting heads is 0.4 for Coin 1 and the probability of getting heads is 0.7 for Coin 2. One of these coins is randomly chosen and then the chosen coin is flipped 10 times.
   (a) What is the probability of getting exactly 7 heads out of the 10 flips?
   (b) Given that the first of these ten flips is heads, what is the conditional probability that exactly 7 out of the 10 flips gives heads?

7. Suppose that the equation $f(x) = 2x^3 + 3x^2 - 11x - 6 = 0$ has at least one integer root. Find all the roots of $f(x)$ by using this information.

8. There are w white balls and b black balls in a box. Balls are drawn one after another without replacement. What is the probability that the white balls will be exhausted before the black balls?

9. Let $f : [0,1] \to [0,1]$ be a strictly convex and continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider a point $x \in (0,1)$. Let A, B, C, and D be the points $(0,0), (1/2,1/2)$, $(1,1)$, and $(x, f(x))$, respectively. What is the ratio between the areas of the triangles $\triangle ABD$ and $\triangle BCD$?

10. Suppose $x^2 + 1 \geq (1+\gamma)x$ is true for all $x \in \mathbb{R}$ if and only if $\gamma \in [a,b]$. Find the values of a and b. Here, $\mathbb{R}$ is the set of all real numbers.