1. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined as follows: $g(x) = f(x) + f(2+x)$, where $f(x) = 1 - |x|$ if $|x| \leq 1$ and $f(x) = 0$ if $|x| > 1$.

   Is the function g differentiable at every point in $\mathbb{R}$? If not, at which points is it not differentiable? **(10)**

2. Consider the following system of equations:

$$
\begin{aligned}
x - y + 3z &= c \\
x + 3y - 3z &= 2c \\
5x + 3y + 3z &= c^2,
\end{aligned}
$$

   where c is some real number.

   (a) Does this system have a unique solution for some values of c? If yes, then find those values of c. **(5)**

   (b) Is this system consistent for some values of c? If yes, then find those values of c. **(5)**

3. (a) It is decided that 10 students will be selected for the interviews based on the JRF QE written test scores. For interviewing purpose, these students are to be *partitioned* into three groups each having at least 2 and at most 4 students. In how many ways can this be done? **(5)**

   (b) Suppose that exactly 5 girls and 5 boys are selected for the interviews. As before, these students are to be partitioned into three groups each having at least 2 and at most 4 students, but now with the additional criteria that there must be at least one girl and at least one boy in each group. In how many ways can this be done? **(5)**

4. Let x_1, x_2, x_3 be three roots of the equation $4x^3 + 3x^2 + 2x + 1 = 0$. Find the value of $x_1^2 + x_2^2 + x_3^2$. **(10)**

5. (a) Consider all triangles such that the length of one side is 1 unit and the area is 1 square unit. Which one of these will have minimum perimeter? In other words, suppose T is the set of all triangles $\triangle ABC$ such that the length of the side AB is 1 unit and the area of $\triangle ABC$ is 1 square unit. Find the length of the sides of a triangle $\triangle XYZ$ in T such that $p(\triangle XYZ) \leq p(\triangle PQR)$ for all $\triangle PQR \in T$. Here, $p(\triangle PQR)$ denotes the perimeter of the triangle $\triangle PQR$. **(5)**

   (b) Consider all triangles having 1 square unit area. Which one of these will have the minimum perimeter? **(5)**

6. State whether the following statements are true or false. If true, provide a proof, and if false, provide a counter example.

(a) Let f and g be two non-negative convex functions on $[0, 10]$. Then, the function $h : [0, 10] \to \mathbb{R}$, defined as $h(x) = f(x)g(x)$ for all $x \in [0, 10]$, is a convex function. **(5)**

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a convex function, then f^2 is also convex. **(5)**

7. A student, E, has missed his Statistics exam. He requests his Statistics teacher, T, to give him some grace marks. T agrees to give him some random score and proposes the following scheme: "I will chose a random number (that is, with uniform probability) from a set S of scores. If the number is bigger than 45, you will get 45. Otherwise, you will get the same score as the chosen random number." When E agrees to this proposal, T asks him to chose one of following two options for S: (i) $S = \{0, 1, 2, \ldots, 100\}$, that is, the set of all integers between 0 and 100, and (ii) $S = [0, 100]$, that is, the set of all real numbers between 0 and 100. E considers both his expected score and the variance of his score to decide an option: (i) if an option gives him higher expected score he will prefer that (regardless of the variance), (ii) if both the options give him the same expected score but one has lower variance than the other, he will prefer the one with lower variance, and (iii) if both the options have the same expectation and variance, he will be indifferent over those.

   What should E's preference be over the two options and why? **(10)**

8. What is the total number of rectangles in a chess board? What is the total number squares in a chess board? (There are 8 rows and 8 columns in a chess board.) **(10)**

9. Consider the matrix $A_{n\times n}$ such that $a_{k,k-1} = a_{k,k+1} = k$ for all $k = 2, 3, \ldots, n-1$, $a_{1,2} = 1$, $a_{n,n-1} = n$, and $a_{i,j} = 0$ for all other cases. What is the rank of $A_{n\times n}$? **(10)**

10. Suppose $\{x_1, x_2, \ldots, x_n\}$ is a set of n positive numbers, where $n \geq 2$. Let the arithmetic mean and the standard deviation be

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}, \text{ respectively.}$$

   Show that the coefficient of variation, defined to be the ratio between standard deviation and mean, that is $s/\bar{x}$, is always strictly less than $\sqrt{n-1}$.
   (**Hint.** You may use the inequality $\sum_{i=1}^{n} x_i^2 < (\sum_{i=1}^{n} x_i)^2$.) **(10)**