1. Suppose that $\{x_n\}_{n\geq 1}$ is a sequence of real numbers such that the inequality $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$ holds for all $n \geq 2$. Prove that $\{x_n\}_{n\geq 1}$ is convergent. [10]

2. Let $\mathbb{Z}$ be the set of all integers. Denote $\gcd(p, q)$ as the greatest common divisior of $p, q \in \mathbb{Z}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{p}{q^2}, & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z},\ q \neq 0 \text{ and gcd(p,q)=1;} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is differentiable at $x = 0$. [10]

3. Show that the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$ has a negative root. [10]

4. Are the vectors $l_1 = (1, 1, 0, -1), l_2 = (5, 5, 0, 0), l_3 = (0, 0, -1, -1)$ linearly independent? [10]

5. Let the random variable $X \sim$ Binomial (n, p). Show that Variance of X is maximum when $p = \frac{1}{2}$. [10]

6. Let X, Y, Z denote 3 jointly distributed random variables with joint density function

$$f(x, y, z) = \begin{cases} K(x^2 + yz), & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find the value of the constant K. Determine the marginal distribution of Y. [6+4]

7. Let x_1 and x_2 be the two real roots of the quadratic equation $ax^2 + ax + c = 0$ where $a \neq 0$ and $c \neq 0$. Moreover, one root is the negative of the reciprocal of the other root, i.e., $x_1 + \frac{1}{x_2} = 0$. Find the values of x_1 and x_2. [10]

8. (a) A code has 4 digits in a specific order. The digits are between $0 - 9$ (including both 0 and 9). How many different codes are possible if each digit may only be used once and we cannot have a code such that 0 is the first digit?

(b) In how many different ways can the word 'political' be rearranged such that the vowels figuring in that word (o, i, i, a) only occur in even places?

(c) In how many different ways can the word 'policy' be rearranged such that the vowels figuring in that word (o, i) only occur in even places?

[2+4+4=10]

9. Consider a consumer in a two-commodity economy whose utility function is given by $U(x_1, x_2) = A - (x_1)^2 - (x_2)^2 + 2a_1x_1 + 2a_2x_2 - (a_1)^2 - (a_2)^2$ with $A > 0$, $a_1 > 0$, $a_2 > 0$, $x_1 \geq 0$ and $x_2 \geq 0$. Let $p_1 > 0$ be the price of commodity 1, $p_2 > 0$ be the price of commodity 2 and M be the money income of the consumer.

(a) If $M > p_1a_1 + p_2a_2$, then what can you say about the optimal value of the Lagragian multiplier associated with the budget constraint $M \geq p_1x_1 + p_2x_2$ when the consumer maximizes his utility subject to the budget constraint? Justify your answer.

(b) If $M < p_1a_1 + p_2a_2$, then how does your result change?

[5+5=10]

10. If a function $f : [0, \infty) \to [0, \infty)$ is concave and $f(0) \geq 0$, then show that $f(x + y) \leq f(x) + f(y)$ for all $x, y \in [0, \infty)$. [10]