1. Answer the following questions. $[7 + 12 + 6 = 25]$

(a) Suppose $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ are two differentiable functions. Suppose $f(x^*) = g(x^*)$ for some $x^* \in (0,1)$ and $f(x) < g(x)$ for all $x < x^*$. Which of the following is true (explain your answer):

(i) $f'(x^*) = g'(x^*)$; (ii) $f'(x^*) > g'(x^*)$; (iii) $f'(x^*) < g'(x^*)$. Here, $f'(x^*)$ and $g'(x^*)$ indicates the derivative of f and g resepctively at x^*.

(b) Assume that $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ are two continuously differentiable functions, i.e., differentiable with continuous derivatives. Suppose $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$ for some $x_1 < x_2$. Show that there is some $x \in [x_1, x_2]$ such that $f'(x) = g'(x)$. Clearly give a statement of the theorem you use to prove your result.

(c) Let $f : [-1,1] \to \mathbb{R}$ and $g : [-1,1] \to \mathbb{R}$ be two functions. Suppose g is continuous and for all $x \in [-1,1]$, we have

$$f(x) = xg(x)$$

Prove that f is differentiable at 0.

2. There are n locations on a street (straight line) situated from left to right as shown in Figure 1.

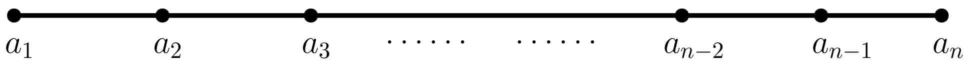

Figure 1: Ordered locations on a street

Let $A = \{a_1, \ldots, a_n\}$ be the set of $n \geq 2$ locations. A consumer's *preference* is a *strict ordering* of locations in A. Denote an arbitrary preference by P and let $a_i \; P \; a_j$ mean that a_i is strictly preferred by consumer than a_j. The highest ranked location according to any preference P is denoted by $P(1)$. Answer the following questions. $[2 + 2 + 3 + 10 + 8 = 25]$

(a) Suppose a consumer can have *any* possible preference over A. How many preferences are possible for the consumer?

(b) A preference P is *single peaked* if for any a_i, a_j

- if a_i is to the right of $P(1)$ and a_j is to the right of a_i, then $a_i \; P \; a_j$
- if a_i is to the left of $P(1)$ and a_j is to the left of a_i, then $a_i \; P \; a_j$

i. Enumerate all single peaked preferences when $A = \{a_1, a_2, a_3, a_4\}$.

ii. Suppose P is a single peaked preference over A and $P(1) = a_k$. What are the possible ranks of a_{k+1}?

iii. Denote by $C(x, y) = \frac{x!}{y!(x-y)!}$ for any non-negative integers $x \geq y$. Show that the number of single peaked preferences over A where top ranked alternative is a_k is $C(n-1, k-1)$.

iv. Use these to show that the number of single peaked preferences over A is 2^{n-1}.

3. Answer the following questions. $[5+5+8+7=25]$

(a) Suppose A and B are defined as

$$A = \min_{x_1, x_2 \in \mathbb{R}_+} \Big(1 - x_1 - 2x_2\Big)^2$$
$$B = \min_{y_1, y_2, y_3 \in \mathbb{R}_+} \Big(1 - y_1 - 2y_2 - 3y_3\Big)^2$$

Which of the following is true? (i) $A = B$ (ii) $A > B$ (iii) $A < B$.

(b) Suppose $f : [0,1] \to \mathbb{R}$ is a differentiable function and let $x^* \in (0,1)$. Suppose x^* satisfies the following property: there exists a $\delta > 0$ such that for all $x \in (x^* - \delta, x^* + \delta)$ we have $f(x) \leq f(x^*)$. Show that the derivative of f at x^* is zero.

(c) Suppose A is a convex set in $\mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is convex if for every $x, y \in A$ and for every $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Show that for each $x_1, \ldots, x_m \in A$ (where $m > 1$) and for each $\lambda_1, \ldots, \lambda_m \in [0,1]$ with $\sum_{i=1}^m \lambda_i = 1$, the following is true:

$$f\Big(\sum_{i=1}^m \lambda_i x_i\Big) \leq \sum_{i=1}^m \lambda_i f(x_i)$$

(d) Let $F : [0,1] \to \mathbb{R}$ be a strictly increasing differentiable function with $F(0) = 0$ and $F(1) = 1$ Consider the following optimization program.

$$\max x(1 - F(x)) \text{ such that } x \in [0, b]$$

where $b \in (0,1)$. Assume $x(1 - F(x))$ is strictly concave. When does the optimal solution x^* of this optimization problem satisfy $x^* = b$?

4. A pair of random variables X_1 and X_2 are jointly distributed with the probability density

$$f(x_1, x_2) = \begin{cases} 8x_1x_2 & \text{if } 0 \leq x_2 \leq x_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Answer the following questions. $[5 + 5 + 5 + 5 + 5 = 25]$

(a) Find the marginal density function of X_1.

(b) Find the marginal density function of X_2.

(c) Show that (marginal) distribution of X_1 first-order stochastic dominates distribution of X_2. In particular, if F_1 and F_2 are marginal cumulative distribution functions of X_1 and X_2 respectively, then show that $F_1(x) \leq F_2(x)$ for all $x \in [0, 1]$.

(d) Show that the expected value of X_1 is no less than the expected value of X_2.

(e) Let Y be the random variable defined by $Y = F_1(X_1)$. Show that Y is uniformly distributed in $[0, 1]$.