1. (a) Consider the consumer demand model given by

$$w_i d\log q_i = b_i(d\log x - \sum_k w_k d\log p_k) + \sum_j c_{ij} d\log p_j$$

where $i = 1, 2, \ldots, n$, q_i is the quantity of i-th item consumed, x is income, p_i is the price of i-th item, $w_i = \frac{p_i q_i}{x}$ is the budget share of item i, and $c_{ij} = \frac{p_i p_j S_{ij}}{x}$, S_{ij} being the compensated price effects, and for any variable z, $d\log z$ denotes total derivative of $\log z$. The parameters are: b_i and $c_{ij}, i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$.

What are the restrictions to be imposed on the parameters for the system to satisfy

(i) Adding-up ($\sum_{i=1}^{n} p_i q_i = x$) and

(ii) Homogeneity of degree zero in prices and income, given (i)?

(b) Derive the Marshallian demand function (in budget share form) for item i for the following indirect utility function:

$$H(x,p) = \sum_{i=1}^{n} a_i \left(\frac{x}{p_i}\right)^{b_i},$$

where $a_i > 0$, $b_i > -1$, and p being the vector of prices.

(c) Derive the Marshallian demand function (in budget share form) for item i for the following cost function:

$$C(u,p) = \sum_{i=1}^{n} p_i a_i + u \prod_{i=1}^{n} p_i^{b_i},$$

where $1 > b_i > 0$ and $\sum_i b_i = 1$.

[(6+6)+6+7]

2. Consider a duopoly market with market demand for the product given by $p = 2 - x_1 - x_2$, where x_i is output of firm i. Firm 1's marginal cost is 1 and this is common knowledge. However, firm 2's marginal cost is determined by Nature. It is $\frac{5}{4}$ with probability $\frac{1}{2}$ and $\frac{3}{4}$ with probability $\frac{1}{2}$. Firms choose quantities simultaneously and non-cooperatively. Find the expected payoff of each firm prior to Nature's move in the following cases:

   (a) At the stage of production neither firm knows firm 2's cost.

   (b) At the stage of production firm 2's cost is private information.

   (c) At the stage of production both firms know each other's costs and this is common knowledge.

[8+10+7]

3. (a) Consider the following regression model

$$y_t = \alpha + e_t$$

   where α is the intercept, $e_t \sim N(0, \sigma^2 x_t)$, $E(e_t e_s) = 0$ for all $t \neq s$ and x_t is non-stochastic and known. Find the best linear unbiased estimator (BLUE) of α, its variance and the estimate of its variance.

   (b) Suppose the true regression model is

$$y_t = \alpha_0 + \alpha_1 x_t + \alpha_2 x_t^2 + v_t.$$

   Suppose instead we estimate the following regression model

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

by ordinary least squares (OLS). Express $E(\hat{\beta}_1)$ in terms of α_0, α_1 and α_2.

[12+13]

4. Consider the following time series model:

$$Y_t = 10 + 0.5Y_{t-2} + \epsilon_t - 0.5\epsilon_{t-2}$$

where ϵ_t's are independent and identically distributed sequence with mean 0 and variance σ^2.

(a) Examine if the time series is stationary or not.

(b) Examine if the time series is invertible or not.

(c) Find the mean and variance of the time series.

[8+7+10]

5. Consider the two sector growth model:

$$Y = AK^{\alpha}H^{\beta}$$
$$\dot{K} = s_K Y - \delta K$$
$$\dot{H} = s_H Y^{\varphi} - \delta H$$

where $\alpha + \beta < 1$, $0 < \varphi \leq 1$, $0 \leq \delta \leq 1$, and $A(t) = A(0)e^{gt}$. Here, Y denotes output, K denotes capital, H denotes human capital, and s_K and s_H are the exogenous savings rates that fund physical and human capital, respectively.

(a) Find the balanced growth path (BGP) for this economy where all variables grow at constant rates (i.e., find the growth rates g_H and g_K in terms of g).

(b) What is the growth rate of output, g_Y, on a BGP?

(c) What is the effect of the parameter, φ, on the growth rate of output on a BGP? Explain your answer.

[10+8+7]

6. Let $\Re, \Re_+$, and $\Re_{++}$ be the sets of real numbers, non negative real numbers, and strictly positive real numbers, respectively. Consider a two commodity economy where the commodity space is $X \times Y$ where $X = Y = \Re_+$. Answer the following questions.

(a) Suppose that the utility function of the consumer is $f(x, y) = \sqrt{xy}$. Show that for any $\lambda \in [0, 1]$ and any $(x_1, y_1), (x_2, y_2) \in \Re^2_{++}$, $f(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \geq \lambda f(x_1, y_1) + (1 - \lambda)f(x_2, y_2)$.

(b) Consider a consumer who makes the following types of purchases.

(i) When the price of commodity 1 is Rs. 5 and that of commodity 2 is Rs. 2, then, for any income level $w > 0$, the consumer buys $\frac{w}{9}$ units of commodity 1 and $\frac{2w}{9}$ units of commodity 2.

(ii) When the price of commodity 1 is Rs. 1 and that of commodity 2 is Rs. 4, then also, for any income level $w > 0$, the consumer buys $\frac{w}{9}$ units of commodity 1 and $\frac{2w}{9}$ units of commodity 2.

If possible, identify a continuous utility function $U : \Re^2_+ \to \Re$, which is non-decreasing in both arguments, for which both of these two types of purchases, are solutions to the consumer's utility maximization problem.

[10+15]

7. Let $\Re, \Re_+$, and $\Re_{++}$ be the sets of real numbers, non negative real numbers, and strictly positive real numbers, respectively. Consider an L-commodity economy and suppose that the commodity space is $X = X_1 \times \ldots \times X_L = \Re^L_+$. Let $\succeq$ be the binary preference relation 'at least as good as' defined on X. Answer the following questions.

   (a) Suppose that the preference relation $\succeq$ on X has a utility representation, that is, there exists a function $U : X \to \Re$ such that for any $x, y \in X$, $x \succeq y$ if and only if $U(x) \geq U(y)$. Then show that the preference relation $\succeq$ on X is rational, i.e., complete and transitive.

   (b) Suppose that the preference relation $\succeq$ on X has a continuous utility function that represents it. Then show that the preference relation $\succeq$ on X is also continuous.

[10+15]

8. Consider a two-factor-two-good economy (say, the Home country) under autarky. Fixed input-output coefficients are given by $a_{L1} = 30$, $a_{L2} = 20$, $a_{K1} = a_{K2} = 1$, where a_{ij} is the amount of the ith factor required to produce one unit of the jth commodity. Endowment of the two factors labour and capital are given by $L = 500$,

$K = 20$. All individuals have an identical utility function given by $U = X_1^{\alpha} X_2^{1-\alpha}, \alpha = 4/7$, where X_1, X_2 are the consumption (and production) of the two commodities under autarky. Find equilibrium levels of output, factor prices, and the relative commodity price under autarky.

Now consider another country (the Foreign country) with identical technology and demand but different endowments. Endowments in the Foreign country are given by $L^* = 700, K^* = 25$. Find equilibrium levels of output, factor prices and the relative commodity price under autarky in the Foreign country. [Hint: Note that both factors cannot be fully employed in autarky in the Foreign country]

Suppose both countries open up to free trade and both countries are price takers (i.e. small) in the world market. Suppose further that in the world market the price of good 1 relative to the price of good 2 is $5/4$. Which good will the Home country export? Which good will the Foreign country export?

[8+9+8]

9. For each of the statements below, prove the result if it is true, give a counter example if it is not true.

   Suppose a game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ has exactly two pure strategy Nash equilibria s and s' such that $u_i(s) \neq u_i(s')$ for all $i \in N$. Then,

   (a) there must be at least two distinct players $i, j \in N$ such that $s_i \neq s'_i$ and $s_j \neq s'_j$,

(b) there is a mixed strategy Nash equilibrium m of G such that $m \notin \{s, s'\}$,

(c) there are infinitely many correlated Nash equilibria of G.

[5+10+10]

10. For each of the statements below, prove the result if it is true, give a counter example if it is not true.

(a) Suppose a game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ does not have any pure strategy Nash equilibrium. Then, for all discount factor $\delta \in (0, 1)$, the two-period repeated game $G^2(\delta)$ of the one-shot game G does not have any pure strategy Nash equilibrium.

(b) For a game $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, let the mixed strategy extension of G with pessimistic players be defined as

$$\hat{G} = \langle N, (M_i)_{i \in N}, (\hat{u}_i)_{i \in N} \rangle,$$

where

- M_i is the set of mixed strategies of player i, and
- for all $m \in \prod_{j \in N} M_j$, $\hat{u}_i(m) = \min \{u_i(s) : s \in \textit{Supp}(m)\}$, where $\textit{Supp}(m) = \{s \in \prod_{j \in N} S_j : m_j(s_j) > 0 \text{ for all } j \in N\}$.

Then, for every game G, the mixed strategy extension $\hat{G}$ of G with pessimistic players has a Nash equilibrium.

[10+15]